SAMPLE PATH LARGE DEVIATIONS FOR QUEUEING NETWORKS WITH BERNOULLI ROUTING

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Abstract. This paper is devoted to the problem of sample path large deviations for multidimensional queueing models with feedback. We derive a new version of the contraction principle where the continuous map is not well-defined on the whole space: we give conditions under which it allows to identify the rate function. We illustrate our technique by deriving a large deviation principle for a class of networks that contains the classical Jackson networks.

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Introduction

This paper is concerned with the theory of large deviations of stochastic processes related to discrete event systems. As opposed to classical stochastic dynamical systems, for which the evolution is continuous and described by a stochastic differential equation, discrete event systems are characterized by synchronization mechanisms that prevent most of the classical tools to apply. We present here a new approach for the analysis of the sample path large deviations of such processes. Unlike standard methods that require establishing upper and lower bounds, our method relies on the analogy between the theory of weak convergence and the theory of large deviations. This analogy is well-known and has been studied by many authors, we refer to the recent book of Feng and Kurtz [11] that surveys this field. We should in particular quote the work of Puhalskii [20], [22] quite similar to our approach. However, we will not use the framework of idempotent measures developed by Puhalskii. We discuss in more details our general methodology and its relation with the existing literature after the description of the queueing networks we consider.

To apply our method we choose a class of queueing networks with Bernoulli routing, where feedback is allowed. The discontinuous dynamic of queueing networks makes it hard to study and large deviations results in the literature are treated on a case by case basis as in the work of Ganesh and Anantharam [12], Bertsimas, Paschalidis and Tsitsiklis [4] or Ramanan and Dupuis [24]. As we will see, adding the possibility of feedback makes the problem much harder. For queueing networks with feedback, existing large deviations results are restricted to networks described by finite-dimensional Markov processes, see the works of Dupuis, Ellis and Weiss [10], Dupuis and Ellis [9] and Ignatiouk-Robert [15]. In this paper, we consider networks where the output process of a queue is modeled by a reflection mapping. This class contains the classical Jackson networks and our large deviations results extend existing results for this class obtained by Atar and Dupuis [3] and Ignatiouk-Robert [14]. Our technique allows to obtain large deviations results under non-exponential assumptions. This case corresponds to networks with autonomous service and gives an approximation for queueing networks where each station acts as a standard single server queue. While preparing this paper, the author became aware of the work of Puhalskii [19] who considers generalized Jackson networks. The form of the rate function for the queue length process obtained in [19] coincides with our result, which confirms the intuition that in the large deviation regime, networks with autonomous service approximate well generalized Jackson networks. We will discuss more carefully this result in Section 2.3.

In the next section, we give an overview of the general methodology and then introduce the general notation. Section 1 gives an extension of the contraction principle that will allow us to identify the rate function. Our result is stated without any reference to any specific discrete event system and could be applied to other systems. In Sections 3 and 4, we apply our method to the case of a queueing network.

General methodology. For simplicity, we adopt here the notation corresponding to our example of queueing network. As in [16] or [17], we define the arrival and departure processes $\mathbf{A}$ and $\mathbf{D}$ of each station of the network as the solution of the fixed point equation

$$\left\{\begin{array}{lcl}\mathbf{A} & = & \Gamma(\mathbf{D},\mathbf{Net}),\\ \mathbf{D} & = & \Phi(\mathbf{A},\mathbf{Net}),\end{array}\right. \quad \Leftrightarrow \quad (\mathbf{A},\mathbf{D}) =: \Psi(\mathbf{Net}). \tag{0.1}$$

Here $\mathbf{Net}$ is a process that describes all the primitives of the network as service times at the different stations, routing decisions, arrival times in the network. The maps Γ and Φ describe the dynamic of the network.

We consider a sequence of queueing networks $\{\mathbf{Net}_n\}_n$, where the primitives are counting processes (i.e. $\mathbf{Net}_n$ belongs to a space denoted by $\mathcal{E}$) and satisfy a large deviation principle (LDP). We denote by $\mathsf{I}^{\mathbf{Net}}$ the corresponding rate function. It is known that the map Ψ is well defined if the primitives of the network are counting processes, see [7] or [16] and we denote $(\mathbf{A}_n, \mathbf{D}_n) = \Psi(\mathbf{Net}_n)$. It is natural to ask whether Ψ is well defined for processes in $\mathbb{D}$, the space of cadlag non-decresing functions or at least for absolutely continuous functions. If this was true, and if Ψ was shown to be continuous, then we would get thanks to the contraction principle that the sequence of processes $\{(\mathbf{A}_n, \mathbf{D}_n)\}_n$ satisfies a LDP with good rate function

$$\text{"}\mathsf{I}^{\mathbf{A},\mathbf{D}}(\mathbf{A}, \mathbf{D}) = \inf\left\{\mathsf{I}^{\mathbf{Net}}(\mathbf{Net}),\ \Psi(\mathbf{Net}) = (\mathbf{A}, \mathbf{D})\right\}.\text{"} \tag{0.2}$$

However, the map Ψ turns out not to be well defined for all possible limits of a sequence of networks $\{\mathbf{Net}_n\}_n \in \mathcal{E}^{\mathbb{N}}$ as defined previously. In particular, the fixed point equation (0.1) can very well be stated for processes in $\mathbb{D}$ but then may have several different solutions as noted by Majewski [17]. We give in the appendix a simple example.

To circumvent this difficulty, we adopt the following strategy. We find a domain $\mathcal{D}_{\mathbf{Net}} \subset \mathcal{E}$ satisfying the following constraints:

- the map Ψ is well defined on $\mathcal{D}_{\mathbf{Net}}$;
- any solution $(\mathbf{A}, \mathbf{D})$ of the fixed point equation (0.1) associated with a "continuous" Jackson network $\mathbf{Net}$ can be approximated by a sequence $\{\mathbf{Net}_n\} \in \mathcal{D}_{\mathbf{Net}}^{\mathbb{N}}$ such that

$$\mathbf{Net}_n \to \mathbf{Net}, \tag{0.3}$$

$$\Psi(\mathbf{Net}_n) \to (\mathbf{A}, \mathbf{D}), \tag{0.4}$$

$$\mathsf{I}^{\mathbf{Net}}(\mathbf{Net}_n) \to \mathsf{I}^{\mathbf{Net}}(\mathbf{Net}). \tag{0.5}$$

Hence in order to remove the quote from (0.2), we follow a quite standard method of proofs for large deviations of stochastic processes analogue with the theory of weak convergence [11]: it consists of first verifying a compactness condition and then showing that there is only one possible limit. In our context, we proceed as follows:

(1) we show that our sequence of processes is exponentially tight;
(2) we use $\mathcal{D}_{\mathbf{Net}}$ to determine the rate function.

In Section 1, we give the theoretical framework that shows how any domain verifying assumptions (0.3), (0.4) and (0.5) determines the rate function. This result is stated in great generality (without any reference to our specific problem) and could be of independent interest since this method of proof could be applied to other dynamical systems (with discontinuous statistics).

Notation. For $(E, d, \leq)$ a complete, separable metric space with partial order $\leq$, we denote by $\mathbb{D}(E)$ the space of cadlag non-decreasing E-valued functions defined on $\mathbb{R}_+$ with Skorohod (J_1) topology and by $\mathbb{C}(E)$ the space of continuous non-decreasing E-valued functions defined on $\mathbb{R}_+$. Restricted to $\mathbb{C}(E)$ the Skorohod topology is just the compact uniform topology.

For $x, y \in \mathbb{R}^K$, we write $x \leq y$ if $x^{(i)} \leq y^{(i)}$ for all i. We denote by $\wedge$ the minimum and by $\vee$ the maximum in $\mathbb{R}^K$. For $\mathbf{X}, \mathbf{Y} \in \mathbb{D}(\mathbb{R}_+^K)$, we write $\mathbf{X} \leq \mathbf{Y}$ if $\mathbf{X}(t) \leq \mathbf{Y}(t)$ for all $t \geq 0$ and for maps $F, G \in \mathbb{D}(\mathbb{R}_+^K)^2$, we denote $F \leq G$ if $F(\mathbf{X}) \leq G(\mathbf{X})$ for all $\mathbf{X} \in \mathbb{D}(\mathbb{R}_+^K)$. For $x \in \mathbb{R}^K$, we denote $\|x\| = \vee_{i=1}^K x^{(i)}$ and for $\mathbf{X} \in \mathbb{D}(\mathbb{R}_+^K)$, we denote $\|\mathbf{X}\| = \sup_t \|\mathbf{X}(t)\|$. We denote $\mathbb{D}_0(E) = \{f \in \mathbb{D}(E),\ f(0) = 0\}$ and $\mathbb{C}_0(E) = \{f \in \mathbb{C}(E),\ f(0) = 0\}$.

A piecewise linear function is a continuous function such that there exists a partition $\tau = (t_0 = 0 < t_1 < \dots)$ with $t_k \to \infty$ and such that the function is linear on each interval (t_k, t_{k+1}).

For any function $f \in \mathbb{D}(\mathbb{R}_+^K)$, we define the polygonal approximation of f with step $1/n$ as the (piecewise linear) function

$$f_n(t) = f\left(\frac{\lfloor nt \rfloor}{n}\right) + (nt - \lfloor nt \rfloor)\left(f\left(\frac{\lfloor nt \rfloor + 1}{n}\right) - f\left(\frac{\lfloor nt \rfloor}{n}\right)\right)$$

$\mathbb{M}^K$ is the set of substochastic matrices of size $K \times K$. For $M \in \mathbb{M}^K$, we denote by $\rho(M)$ its spectral radius, by M^t its transpose and $M^{(i)}$ denotes the line $M^{(i)} = (M^{(i,1)}, \dots M^{(i,K)})$. In particular, we will identify a function $\mathbf{P} \in \mathbb{D}(\mathbb{M}^K)$ with its K components $\mathbf{P}^{(i)} \in \mathbb{D}(\mathbb{R}_+^K)$, where $\mathbf{P}^{(i)}(t) = (\mathbf{P}^{(i,1)}(t), \dots \mathbf{P}^{(i,K)}(t))$ with $\sum_j \mathbf{P}^{(i,j)}(t) \leq 1$ for all $t \geq 0$ and all i. Note that for $M, N \in \mathbb{M}^K$, we have $M \leq N$ if $M^{(i,j)} \leq N^{(i,j)}$ for all i and j.

We will use the Kullback-Leibler information divergence, which is a nonsymmetric measure of distance between distributions in the sense that for any two distributions P and R on $\mathcal{X}^k$ where $\mathcal{X}$ is a finite set,

$$\mathsf{D}(P\|R) = \sum_{x \in \mathcal{X}^k} P(x) \log\left(\frac{P(x)}{R(x)}\right),$$

is nonnegative and equals 0 if and only if $P = R$. We use the standard notational conventions $\log 0 = -\infty$, $\log \frac{1}{0} = \infty$ and $0 \log 0 = 0 \log \frac{0}{0} = 0$. For any fixed R, the divergence $\mathsf{D}(P\|R)$ is a continuous function of P restricted to $\{P,\ S(P) \subset S(Q)\}$ where $S(P)$ denotes the support of P (see [8]).

For $P \in \mathbb{M}^K$, we denote by $\tilde{P}$ the $K \times (K+1)$ stochastic matrix obtained as follows: for all $i, j \leq K$, $\tilde{P}^{(i,j)} = P^{(i,j)}$ and $\tilde{P}^{(i,K+1)} = 1 - \sum_{k=1}^K P^{(i,k)}$. For $P, R \in \mathbb{M}^K$, we will denote

$$\begin{aligned}
\tilde{\mathsf{D}}(P\|R) &:= \mathsf{D}(\tilde{P}\|\tilde{R}) \\
&= \sum_{i,j \leq K} P^{(i,j)} \log\left(\frac{P^{(i,j)}}{R^{i,j)}}\right) + \sum_{i \leq K}\left(1 - \sum_k P^{(i,k)}\right) \log\left(\frac{1 - \sum_k P^{(i,k)}}{1 - \sum_k R^{(i,k)}}\right) \\
&=: \sum_{i=1}^K \tilde{\mathsf{D}}(P^{(i)}\|R^{(i)}).
\end{aligned}$$

1. An extension of the contraction principle

Let $\mathcal{E}, \mathcal{F}$ be complete separable metric spaces. Let $G : \mathcal{E} \times \mathcal{F} \to \mathbb{R}$ be a continuous function. We assume that there exists $\mathcal{D} \subset \mathcal{E}$, such that for all $x \in \mathcal{D}$, there exists an unique $y \in \mathcal{F}$ such that $G(x, y) = 0$. We denote it by, $y = H(x)$ where $H : \mathcal{D} \to \mathcal{F}$,

$$\forall x \in \mathcal{D}, \quad G(x, y) = 0 \ \Leftrightarrow \ y = H(x).$$

Proposition 1.1. *Let $\{X_n\}_n$ be a sequence of $\mathcal{E}$-valued random variables and $\{Y_n\}_n$ be a sequence of $\mathcal{F}$-valued random variables. We assume that each sequence is exponentially tight. Assume that the sequence $\{X_n\}_n$ satisfies a LDP with good rate function I^X and that $G(X_n, Y_n) = 0$ a.s. for all n.*

We assume that for all (x, y) such that $G(x, y) = 0$ and $\mathsf{I}^X(x) < \infty$, there exists a sequence $x_n \to x$, such that $x_n \in \mathcal{D}$ for all n, $H(x_n) \to y$ and $\mathsf{I}^X(x_n) \to \mathsf{I}^X(x)$. We denote by $\mathcal{S}(x, y) = \{x_n\}_n$ this sequence. If $G(x, y) \neq 0$ or $\mathsf{I}^X(x) = \infty$, we take $\mathcal{S}(x, y) = \emptyset$ and we denote $\mathcal{S}(y) = \cup_x\{\mathcal{S}(x, y)\}$ (which might be empty).

Then the sequence $\{X_n, Y_n\}_n$ *satisfies a LDP with good rate function:*

$$\mathsf{I}^{X,Y}(x,y) := \begin{cases} \mathsf{I}^X(x), & G(x,y)=0, \\ \infty, & \textit{otherwise}. \end{cases} \tag{1.1}$$

In particular, if $X_n \in \mathcal{D}$ *for all* n *and if the sequence* $\{H(X_n)\}_n$ *is exponentially tight, then it satisfies a LDP in* $\mathcal{F}$ *with good rate function:*

$$\mathsf{I}^{H(X)}(y) := \inf\{\lim_{n\to\infty} \mathsf{I}^X(x_n),\ \{x_n\}_n \in \mathcal{S}(y)\}. \tag{1.2}$$

Remark 1.1.

- There are alternative ways of expressing the rate function,
$$\mathsf{I}^{H(X)}(y) = \inf\{\mathsf{I}^X(x),\ y \in H^x\},$$
where $H^x := \{y \in \mathcal{F},\ \exists x_n \to x,\ H(x_n) \to y\}$. $\mathsf{I}^{H(X)}$ is the lower semicontinuous regularization of the following function defined for $y \in H(\mathcal{D}) \subset \mathcal{F}$,
$$\tilde{\mathsf{I}}^{H(X)}(y) := \inf\{\mathsf{I}^X(x),\ y = H(x)\}.$$
The main interest of the definition (1.2) is that the rate function is computed only thanks to the sequences $\mathcal{S}(x,y) \in \mathcal{D}^{\mathbb{N}}$.
- Note that if $H(\mathcal{D})$ is closed (in particular if $\mathcal{D} = \mathcal{E}$) then this proposition follows from the contraction principle (for an extensive discussion of this principle, see the work of Garcia [13]). Roughly speaking, Proposition 1.1 tells us that if $\mathcal{D}$ is dense in a certain sense in $\mathcal{E}$, then the contraction principle still holds for the map H.

Proof. Thanks to Lemma 3.6 of [11], the sequence $\{X_n, Y_n\}_n$ is exponentially tight. Then by Theorem 3.7 of [11], there exists a subsequence $\{n_k\}$ along which the sequence $\{X_{n_k}, Y_{n_k}\}_{n_k}$ satisfies a LDP with a good rate function. If we can prove that there is a unique possible rate function (that does not depend on the subsequence $\{n_k\}$) then the proposition will follow.

Hence, for simplicity of notations, we still denote the extracted subsequence by $\{X_n, Y_n\}_n$ and we assume that $\{X_n, Y_n\}_n$ satisfies a LDP with good rate function $\tilde{\mathsf{I}}^{X,Y}$. We will show that $\tilde{\mathsf{I}}^{X,Y} = \mathsf{I}^{X,Y}$ given by (1.1).

Consider the continuous mappings H_1 and H_2 from $\mathcal{E} \times \mathcal{F}$ to $\mathcal{E} \times \mathcal{F} \times \mathbb{R}$,

$$H_1(x,y) := (x, y, G(x,y)), \quad H_2(x,y) := (x,y,0).$$

We have clearly $H_1(X_n, Y_n) = H_2(X_n, Y_n)$ a.s. Moreover thanks to the contraction principle, $\{H_1(X_n, Y_n)\}_n$ and $\{H_2(X_n, Y_n)\}_n$ satisfy LDPs with the good rate functions

$$\mathsf{I}^{H_1}(x,y,z) = \inf\{\tilde{\mathsf{I}}^{X,Y}(x,y), z = G(x,y)\} \quad \mathsf{I}^{H_2}(x,y,z) = \inf\{\tilde{\mathsf{I}}^{X,Y}(x,y), z = 0\},$$

where $\inf \emptyset = \infty$. Since $H_1(X_n, Y_n) = H_2(X_n, Y_n)$, we have $\mathsf{I}^{H_1} = \mathsf{I}^{H_2}$. Now we have,

$$\tilde{\mathsf{I}}^{X,Y}(x,y) = \inf_z\{\mathsf{I}^{H_1}(x,y,z)\} = \inf\{\tilde{\mathsf{I}}^{X,Y}(x,y), G(x,y) = 0\},$$

hence $\tilde{\mathsf{I}}^{X,Y}(x,y) = \infty$ as soon as $G(x,y) \neq 0$. It remains to show that $G(x,y) = 0$ implies $\tilde{\mathsf{I}}^{X,Y}(x,y) = \mathsf{I}^X(x)$. We have clearly $\mathsf{I}^X(x) \le \tilde{\mathsf{I}}^{X,Y}(x,y)$ for all (x,y) since $\{X_n\}$ satisfies a LDP with good rate function

$$\mathsf{I}^X(x) = \inf\{\tilde{\mathsf{I}}^{X,Y}(x,y),\ y \in \mathcal{F},\ G(x,y) = 0\}.$$

In particular, the definition of $\mathcal{D}$ implies $\mathsf{I}^X(x) = \tilde{\mathsf{I}}^{X,Y}(x, H(x))$ for $x \in \mathcal{D}$.

Take (x,y) such that $G(x,y)=0$ and $\mathsf{I}^X(x)<\infty$. There exists $x_n^*\to x$ with $x_n^*\in\mathcal{D}$, $H(x_n^*)\to y$ and $\mathsf{I}^X(x_n^*)\to\mathsf{I}^X(x)$. Thanks to the lower semicontinuity property of $\tilde{\mathsf{I}}^{X,Y}$, we can find for any $\delta>0$, an $\epsilon>0$ such that

$$\frac{1}{\delta}\wedge\left(\tilde{\mathsf{I}}^{X,Y}(x,y)-\delta\right)\leq\inf_{z\in B(y,\epsilon)}\tilde{\mathsf{I}}^{X,Y}(x,z),$$

where $B(y,\epsilon)$ is the closed ball in $\mathcal{F}$ of center y and radius ϵ.

Thanks to the lower semicontinuity of the function $x\mapsto\inf_{z\in B(y,\epsilon)}\tilde{\mathsf{I}}^{X,Y}(x,z)$, we have

$$\begin{aligned}\inf_{z\in B(y,\epsilon)}\tilde{\mathsf{I}}^{X,Y}(x,z) &\leq \liminf_{x_n\to x}\inf_{z\in B(y,\epsilon)}\tilde{\mathsf{I}}^{X,Y}(x_n,z)\\ &\leq \liminf_{n\to\infty}\inf_{z\in B(y,\epsilon)}\tilde{\mathsf{I}}^{X,Y}(x_n^*,z)\\ &\leq \lim_{n\to\infty}\mathsf{I}^X(x_n^*)=\mathsf{I}^X(x),\end{aligned}$$

because $H(x_n^*)\in B(y,\epsilon)$ for sufficiently large n. Hence we proved that for any $\delta>0$, $\frac{1}{\delta}\wedge\left(\tilde{\mathsf{I}}^{X,Y}(x,y)-\delta\right)\leq\mathsf{I}^X(x)$ for (x,y) such that $G(x,y)=0$ and $\mathsf{I}^X(x)<\infty$, this concludes the proof of (1.1).

The various expressions of $\mathsf{I}^{H(X)}$ are now quite easy to obtain from

$$\mathsf{I}^{H(X)}(y)=\inf\{\mathsf{I}^X(x),\ G(x,y)=0\}. \tag{1.3}$$

For (1.2), note that since the set $\{x,\ G(x,y)=0\}$ is closed the minimum in (1.3) (if it is finite) is attained for a certain x^* with $G(x^*,y)=0$ and $\mathsf{I}^X(x^*)<\infty$.

We prove now that

$$\inf\{\mathsf{I}^X(x),\ y\in H^x\}=\inf\{\mathsf{I}^X(x),\ G(x,y)=0\}.$$

If $y\in H^x$, then there exists $x_n\to x$ such that $H(x_n)\to y$. Hence by continuity of G, we have $G(x,y)=0$. Now if $G(x,y)=0$ and $\mathsf{I}^X(x)<\infty$, it follows from the assumptions that $y\in H^x$.

To see that the last expression in Remark 1.1 is true, we show that for any open set $O\subset\mathcal{F}$, we have,

$$\inf_{y\in O}\mathsf{I}^{H(X)}(y)=\inf_{y\in O}\{\mathsf{I}^X(x),\ y=H(x)\}. \tag{1.4}$$

For $y\in O$ and any x such that $G(x,y)=0$, there exists $x_n\to x$, such that $H(x_n)\to y$ and $\mathsf{I}^X(x_n)\to\mathsf{I}^X(x)$. Hence for n sufficiently large, we have $H(x_n)\in O$ and then

$$\inf_{y\in O}\{\mathsf{I}^X(x),\ y=H(x)\}\leq\inf_n\mathsf{I}^X(x_n)\leq\mathsf{I}^X(x).$$

Taking the minimum over all x such that $G(x,y)=0$ gives the $\geq$ inequality in (1.4), the converse inequality is obvious. □

2. Queueing networks with Bernoulli routing: description and large deviations results

2.1. **General setting and notation.** We start with the basic model for an isolated queue and refer to [1] for more details on the relationship with other models of the literature.

The model for an isolated queue is in term of two primitive quantities belonging to $\mathbb{D}(\mathbb{R}_+)$: the arrival process $\mathbf{A}$ and the service process $\mathbf{S}$. The departure process $\mathbf{D}$ is a derived quantity that is obtained as a functional of the arrival and service processes as follows:

$$\mathbf{D}(t) := \inf_{0\le s\le t} \{\mathbf{S}(t) - \mathbf{S}(s) + \mathbf{A}(s)\} \wedge \mathbf{S}(t). \tag{2.1}$$

From a mathematical point of view, if $\mathcal{R} : \mathbb{D} \to \mathbb{D}$ (where $\mathbb{D}$ is the space of cadlag $\mathbb{R}$-valued functions defined on $\mathbb{R}_+$) is the one-dimensional Skorohod's reflection map defined by $\mathcal{R}(\mathbf{X})(t) := \sup_{0\le s\le t} \{\mathbf{X}(t) - \mathbf{X}(s)\} \vee \mathbf{X}(t)$. We have $\mathbf{D} = \mathbf{A} - \mathcal{R}(\mathbf{A} - \mathbf{S})$. It is easy to see that $\mathbf{D} \in \mathbb{D}(\mathbb{R}_+)$ and $\mathbf{D} \le \mathbf{A}$.

The queue length process is defined as the difference of the arrival process and the departure process,

$$\mathbf{Q}(t) := \mathbf{A}(t) - \mathbf{D}(t) = \sup_{0\le s\le t} \{\mathbf{A}(t) - \mathbf{A}(s) - (\mathbf{S}(t) - \mathbf{S}(s))\} \vee (\mathbf{A}(t) - \mathbf{S}(t)).$$

If the arrival process $\mathbf{A}$ and the service process $\mathbf{S}$ are counting processes, this model is called a single queue with autonomous service: the queue length is increased by one whenever there is an arrival from the arrival process and the queue length is decreased by one whenever there is an arrival from the service process and the queue is not empty (see [5]). Note in particular that in the case where the process $\mathbf{S}$ is a Poisson point process, then this model is a standard $./M/1$ queue.

We now consider networks obtained by interconnecting queues modeled by (2.1) when the departure process of one queue is randomly routed to the other queues as for Jackson networks. The networks we consider are characterized by the fact that service times and routing decisions are associated with stations and not with customers. This means that we associate to each of the K stations three predefined counting processes: an arrival process, a service process and a routing process. The arrival process and the service process of station k are described by the sequences of exogenous arrival times $\{T_j^{(k)}\}_{j\ge 1}$ and service times $\{\sigma_j^{(k)}\}_{j\ge 1}$. If there is no exogenous arrival at station k, we use the convention $T_j^{(k)} = \infty$ for all j. When the j-th customer has completed his service at station k, he is sent to station $\nu_j^{(k)}$ (or leaves the network if $\nu_j^{(k)} = K+1$) and is put at the end of the queue on this station, where $\{\nu_j^{(k)}\}_{j\ge 1}$ is also a predefined sequence, called the routing sequence. The sequences $\{T_j^{(k)}\}_{j\ge 1}$, $\{\sigma_j^{(k)}\}_{j\ge 1}$ and $\{\nu_j^{(k)}\}_{j\ge 1}$, where k ranges over the set of stations, are called the driving sequences of the network. A network will be defined by $\left\{\{\sigma_j^{(k)}\}_{j\ge 1},\ \{\nu_j^{(k)}\}_{j\ge 1},\ \{T_j^{(k)}\}_{j\ge 1},\ n^{(k)},\ 1\le k\le K\right\}$, where $(n^{(1)},\dots,n^{(K)})$ describes the initial condition. The interpretation is as follows: at time $t = 0$, in node k, there are $n^{(k)}$ customers with service times $\sigma_1^{(k)},\dots,\sigma_{n^{(k)}}^{(k)}$ (if appropriate, $\sigma_1^{(k)}$ may be interpreted as a residual service time). In particular at time 0, the total number of customers in the network is $n^{(1,K)} = n^{(1)} + \dots n^{(K)}$.

In what follows, we will describe the driving sequences thanks to their associated counting functions. We will use the following notation: $\sigma^{(k)}(1,n) = \sum_{j=1}^{n} \sigma_j^{(k)}$, for $0 \le k \le K$.

We define the sequence of networks $\mathbf{Net}_n = \{\mathbf{S}_n(t), \mathbf{P}_n(t), \mathbf{N}_n(t)\}$ with

$$\begin{aligned}
\mathbf{N}_n^{(i)}(t) &= \frac{1}{n}\left(n_n^{(i)} + \sum_k \mathbb{1}_{\{T_k^{(i)} \le nt\}} \right), \\
\mathbf{S}_n^{(i)}(t) &= \frac{1}{n}\sum_k \mathbb{1}_{\{\sigma^{(i)}(1,k) \le nt\}}, \\
\mathbf{P}_n^{(i,j)}(t) &= \frac{1}{n}\sum_{k \le nt} \mathbb{1}_{\{\nu_k^{(i)} = j\}}.
\end{aligned}$$

Note that we allow the initial queue length to depend on n, $\mathbf{N}_n^{(i)}(0) = n_n^{(i)}$ but the other driving sequences describing the arrival times, the service times and the routing decisions do not depend on n. Note also that if there is no exogenous arrival at station i, we have $\mathbf{N}_n^{(i)}(t) = \mathbf{N}_n^{(i)}(0)$ for all $t \ge 0$.

For the network $\mathbf{Net}_n$, we denote the corresponding input and output processes of each queue k of the network by $\mathbf{A}_n^{(k)}$ and $\mathbf{D}_n^{(k)}$ respectively. We will use the following notation $\mathbf{A}_n = (\mathbf{A}_n^{(1)}, \dots, \mathbf{A}_n^{(K)})$ and $\mathbf{D}_n = (\mathbf{D}_n^{(1)}, \dots, \mathbf{D}_n^{(K)})$. We now describe how the processes $\mathbf{A}_n$ and $\mathbf{D}_n$ are obtained form $\mathbf{Net}_n$.

We define the map $\Gamma : \mathbb{D}_0(\mathbb{R}_+^K) \times \mathbb{D}_0(\mathbb{M}^K) \times \mathbb{D}(\mathbb{R}_+^K) \to \mathbb{D}(\mathbb{R}_+^K)$ as follows:

$$\Gamma(\mathbf{X}, \mathbf{P}, \mathbf{N})^{(i)}(t) := \mathbf{N}^{(i)}(t) + \sum_{j=1}^K \mathbf{P}^{(j,i)}(\mathbf{X}^{(j)}(t)).$$

The following lemma is straightforward.

Lemma 2.1. *The map Γ is continuous for the compact uniform topology and non-decreasing in its first argument.*

We define the map $\Phi : \mathbb{D}(\mathbb{R}_+^K) \times \mathbb{D}_0(\mathbb{R}_+^K) \to \mathbb{D}_0(\mathbb{R}_+^K)$ as follows:

$$\Phi(\mathbf{X}, \mathbf{Y})^{(i)}(t) := \inf_{0 \le s \le t} \left\{ \mathbf{Y}^{(i)}(t) - \mathbf{Y}^{(i)}(s) + \mathbf{X}^{(i)}(s) \right\} \wedge \mathbf{Y}^{(i)}(t).$$

Lemma 2.2. *The map Φ is continuous for the compact uniform topology and non-decreasing in its first argument.*

Proof. We can clearly consider the map Φ with $K = 1$ only. Let $\mathcal{R}$ be the one-dimensional reflection map, we have $\Phi(\mathbf{X}, \mathbf{Y}) = \mathbf{X} - \mathcal{R}(\mathbf{X} - \mathbf{Y})$. It is easy to see that for any $T > 0$,

$$\sup_{0 \le t \le T} |\mathcal{R}(\mathbf{X})(t) - \mathcal{R}(\mathbf{X}')(t)| \le 2 \sup_{0 \le t \le T} |\mathbf{X}(t) - \mathbf{X}'(t)|,$$

from which the continuity of Φ follows. Its monotonicity is obvious. $\square$

Remark 2.1. Consider the mapping Φ with $K = 1$ and $\mathbf{Y}(t) = \mu t$, with $\mu \ge 0$. If $\mu = 0$, since $\Phi(\mathbf{X}, \mathbf{Y}) \le \mathbf{Y}$, we have $\Phi(\mathbf{X}, \mathbf{Y})(t) = 0$ for all t. If $\mu \ne 0$, we have $\Phi(\mathbf{X}, \mathbf{Y})(t) = \inf_{0 \le s \le t}\{\mathbf{X}(s) + \mu(t-s)\}$. Moreover if $\mathbf{X}$ is a concave function, then this equation reduces to $\Phi(\mathbf{X}, \mathbf{Y})(t) = \mathbf{X}(t) \wedge \mu t$. Hence we can write

$$\begin{aligned}
\mathbf{Y}(t) = \mu t, \text{ with } \mu \ge 0 \quad &\Rightarrow \quad \Phi(\mathbf{X}, \mathbf{Y})(t) = \mu t \wedge \inf_{0 \le s \le t}\{\mathbf{X}(s) + \mu(t-s)\}, \\
\text{if moreover } \mathbf{X} \text{ is a concave function} \quad &\Rightarrow \quad \Phi(\mathbf{X}, \mathbf{Y})(t) = \mu t \wedge \mathbf{X}(t).
\end{aligned}$$

It is easy to adapt the proof of Theorem 2.1 of [7] or Proposition 2.1 of [16] to show that the following fixed-point equation:

$$\left\{ \begin{array}{lcl} \mathbf{A}_n & = & \Gamma(\mathbf{D}_n, \mathbf{P}_n, \mathbf{N}_n) = \Gamma(\mathbf{D}_n, \mathbf{Net}_n), \\ \mathbf{D}_n & = & \Phi(\mathbf{A}_n, \mathbf{S}_n) = \Phi(\mathbf{A}_n, \mathbf{Net}_n), \end{array} \right. \tag{2.2}$$

has an unique solution when each component of $n\mathbf{S}_n$, $n\mathbf{P}_n$ and $n\mathbf{N}_n$ is a counting function (i.e. non-decreasing function of $\mathbb{D}(\mathbb{R}_+^K)$ or $\mathbb{D}(\mathbb{M}^K)$ that is piece-wise constant with jumps of size one). In this case the corresponding functions $n\mathbf{A}_n$ and $n\mathbf{D}_n$ are also counting functions and we denote the solution of (2.2) by $\Psi(\mathbf{S}_n, \mathbf{P}_n, \mathbf{N}_n) = \Psi(\mathbf{Net}_n)$.

Remark 2.2. Note that the only difference between our model and generalized Jackson networks as described in [16] resides in the queueing mechanism (2.1) which is sometimes called autonomous. Consider a network $\mathbf{Net} = \{\mathbf{S}, \mathbf{P}, \mathbf{N}\}$ where the processes are counting processes. Then due to some monotonicity arguments, it is possible to relate (see [6]):
- the process $(\tilde{\mathbf{A}}, \tilde{\mathbf{D}})$ associated to $\mathbf{Net}$ with the dynamic described in [16];
- the processes $\Psi(\mathbf{Net}) = (\mathbf{A}, \mathbf{D})$ solution of the fixed point equation.
Note that in the case where the process $\mathbf{S}$ is a Poisson point process, our model is exactly a Jackson network (see [2]).

2.2. **Stochastic assumptions.** In what follows, it will be important to distinguish the nodes of the network that do not receive any exogenous customer, i.e. the nodes $i \in \mathcal{S}^c$ with $\mathcal{S} = \{i,\ T_1^{(i)} \leq \infty\}$. A network $\mathbf{Net} = \{\mathbf{S}, \mathbf{P}, \mathbf{N}\}$ is an object in $\mathcal{E} \subset \mathbb{D}_0(\mathbb{R}_+^K) \times \mathbb{D}_0(\mathbb{M}^K) \times \mathbb{D}(\mathbb{R}_+^K)$, with the additional constraints:

(1) $\mathbf{N}^{(i)}(t) = \mathbf{N}^{(i)}(0)$ for all t, for $i \notin \mathcal{S}$;
(2) for all $0 \leq v \leq u$, we have $\sum_{j=1}^K \mathbf{P}^{(i,j)}(u) - \mathbf{P}^{(i,j)}(v) \leq (u - v)$.

Note that $\mathcal{E}$ is closed in $\mathbb{D}_0(\mathbb{R}_+^K) \times \mathbb{D}_0(\mathbb{M}^K) \times \mathbb{D}(\mathbb{R}_+^K)$.

We define for $(s^{(1)}, \dots, s^{(K)}) \in \mathbb{R}_+^K$ and $(n^{(1)}, \dots, n^{(K)}) \in \mathbb{R}_+^K$, the functions

$$\begin{array}{rcl} \mathsf{I}^{\mathbf{S}}(s^{(1)}, \dots, s^{(K)}) & = & \displaystyle\sum_{i=1}^K \mathsf{I}^{\mathbf{S}^{(i)}}(s^{(i)}), \\ \mathsf{I}^{\mathbf{N}}(n^{(1)}, \dots, n^{(K)}) & = & \displaystyle\sum_{i \in \mathcal{S}} \mathsf{I}^{\mathbf{N}^{(i)}}(n^{(i)}) + \infty \mathbb{1}_{\{n^{(i)} > 0,\ i \notin \mathcal{S}\}}, \end{array}$$

where each $\mathsf{I}^{\mathbf{S}^{(i)}}$ (resp. $\mathsf{I}^{\mathbf{N}^{(i)}}$ for $i \in \mathcal{S}$) is a $[0, \infty]$-valued convex good rate function, attaining zero on $\mathbb{R}_+$ admitting a unique minimum at the point $\mu^{(i)}$ (resp. $\lambda^{(i)}$ for $i \in \mathcal{S}$) and with a domain open on the right.

We assume that the sequence $\mathbf{Net}_n = \{\mathbf{S}_n(t), \mathbf{P}_n(t), \mathbf{N}_n(t)\}$ satisfies a LDP in the space $\mathcal{E}$ with a good rate function $\mathsf{I}^{\mathbf{Net}}$ given by

$$\mathsf{I}^{\mathbf{Net}}(\mathbf{S}, \mathbf{P}, \mathbf{N}) := \mathsf{I}^0(\mathbf{N}(0)) + \int_0^\infty \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t)) + \tilde{\mathsf{D}}(\dot{\mathbf{P}}(t) \| R) + \mathsf{I}^{\mathbf{N}}(\dot{\mathbf{N}}(t)) dt, \tag{2.3}$$

if the argument functions are absolutely continuous and equal to infinity otherwise.

We make the following assumptions on the matrix R:

(1) We assume that $\rho(R) < 1$.

(2) We assume that for all $1 \leq i \leq K$, we have

$$(\mathcal{N} + \mathcal{N}R + \cdots + \mathcal{N}R^K)^{(i)} > 0, \tag{2.4}$$

where $\mathcal{N}$ is the line vector of $\mathbb{R}_+^K$ defined by $\mathcal{N}^{(i)} = \mathbb{1}_{\{i \in \mathcal{S}\}}$.

We show now that our stochastic assumptions cover the case where $\mathbf{S}^{(k)}$ and $\mathbf{N}^{(i)}$ (with $i \in \mathcal{S}$) are independent and correspond to renewal processes and where the routing is a Bernouilli routing associated with the matrix R that satisfies previous assumption.

We recall here some results of Puhalskii [21] concerning large deviations of renewal processes and show that our assumptions on the rate function (2.3) are satisfied in the i.i.d case. Denote by $\{\zeta_i,\ i \geq 1\}$ a sequence of non-negative i.i.d. random variables with positive mean. Let

$$\begin{aligned}
\alpha(\theta) &= \log \mathbb{E}\left[e^{\theta\zeta_1}\right], \\
\theta^* &= \sup\{\theta > 0,\ \alpha(\theta) < \infty\}, \\
\alpha^*(x) &= \sup_\theta\{\theta x - \alpha(\theta)\} = \sup_{\theta<\theta^*}\{\theta x - \alpha(\theta)\}, \\
g(x) &= x\alpha^*(1/x) = \sup_{\theta<\theta^*}\{\theta - x\alpha(\theta)\}.
\end{aligned} \tag{2.5}$$

Note that the function α is a convex function and differentiable on $(-\infty, \theta^*)$ with $\alpha'(0) = \mathbb{E}[\zeta_1] > 0$. In particular, we have $\lim_{\theta\uparrow\theta^*} \alpha(\theta) = \infty$, from which we get the equality in (2.5). The functions α^* and g are convex rate functions. Introduce the sequence of processes $\{\mathbf{C}_n\}_n$:

$$\mathbf{C}_n(t) = \frac{1}{n}\sum_i \mathbb{1}_{\{\sum_{j=1}^i \zeta_j \leq nt\}}.$$

Then Theorem 3.1 of [21] gives: If $\mathbb{P}(\zeta_1 > 0) = 1$, then the sequence $\{\mathbf{C}_n\}_n$ satisfies a LDP in $\mathbb{D}(\mathbb{R}_+)$ with the good rate function

$$\mathsf{I}^{\mathbf{C}}(\mathbf{x}) = \begin{cases} \int_0^\infty g(\dot{\mathbf{x}}(t))dt, & \text{if } \mathbf{x} \in \mathbb{C}(\mathbb{R}_+) \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

It then follows that g is a good rate function. Moreover, we have $\operatorname{essinf}\zeta_1 = 0$ if and only if $g(x)$ is finite for all $x \geq \mathbb{E}[\zeta_1]^{-1}$ (note in particular, that in this case, the domain of g is open on the right). The proof of this fact can be found in [19] and follows the argument: from $\alpha(\theta) \geq \mathbb{E}[\zeta_1]\theta$, we have $g\left(\mathbb{E}[\zeta_1]^{-1}\right) = 0$ and for all $x \geq \mathbb{E}[\zeta_1]^{-1}$, we have $g(x) = \sup_{\theta\leq 0}\{\theta - x\alpha(\theta)\}$. If $\operatorname{essinf}\zeta_1 = 0$, we have for arbitrary $\epsilon > 0$ and for $\theta \leq 0$,

$$\alpha(\theta) = \log \mathbb{E}\left[e^{\theta\zeta_1}\right] \geq \theta\epsilon + \log \mathbb{P}(\zeta_1 < \epsilon),$$

hence for $x > \epsilon^{-1}$, we have $g(x) \leq -\log \mathbb{P}(\zeta_1 < \epsilon)$. It is clear that if $\operatorname{essinf}\zeta_1 > 0$, then for any $x > \operatorname{essinf}\zeta_1^{-1}$, we have $g(x) = \infty$.

Concerning the large deviations of the routing processes given in term of the Kullback-Leibler information divergence, it follows directly from Corollary 6.1 of [23] in the case of Bernouilli routing, i.e. when the sequences $\{\nu_j^{(k)}\}_{j\geq 1}$ are sequences of i.i.d. random variables in $[1, K]$ and independent in k such that

$$\mathbb{P}(\nu_1^{(k)} = i) = R^{(k,i)}.$$

2.3. **Sample path large deviations for the queue length process.** We now return to the sequence of queueing networks defined in Section 2. Recall that $(\mathbf{A}_n, \mathbf{D}_n)$ correspond to the arrival and departure processes from each station. We now give our theorem for the queue length process defined as $\mathbf{Q}_n = \mathbf{A}_n - \mathbf{D}_n$.

Theorem 2.1. *The sequence of processes $\{\mathbf{Q}_n\}_n$ satisfies a LDP in $\mathbb{D}(\mathbb{R}_+^K)$ with good rate function that is finite for $\mathbf{Q}$ absolutely continuous given by:*

$$\mathsf{I}^0(\mathbf{Q}(0)) + \mathsf{I}^{\mathbf{Q}}_{\mathbf{Q}(0)}(\mathbf{Q}),$$

where for $q \geq 0$, $\mathsf{I}^{\mathbf{Q}}_q(.)$ is a good rate function that is finite for absolutely continuous $\mathbf{Q}$ such that $\mathbf{Q}(0) = q$ and given by:

$$\mathsf{I}^{\mathbf{Q}}_q(\mathbf{Q}) := \int_0^\infty \mathsf{H}^{\mathbf{Q}}(\mathbf{Q}(s), \dot{\mathbf{Q}}(s))ds,$$

where $\mathsf{H}^{\mathbf{Q}}$ is given by,

$$\mathsf{H}^{\mathbf{Q}}(Q, \dot{Q}) := \inf \left\{ \sum_{i \in E(Q)} \mathsf{I}^{\mathbf{S}^{(i)}}(D^{(i)}) \mathbb{1}_{\{D^{(i)} > \mu^{(i)}\}} + \sum_{i \notin E(Q)} \mathsf{I}^{\mathbf{S}^{(i)}}(D^{(i)}) + \sum_i D^{(i)} \tilde{\mathsf{D}}(P^{(i)} \| R^{(i)}) + \mathsf{I}^{\mathbf{N}}(N) \right\}$$

where $E(Q) = \{i, Q^{(i)} = 0\}$ and the infimum is taken over the set of $(D, P, N) \in \mathbb{R}_+^K \times \mathbb{M}^K \times \mathbb{R}_+^K$ such that

$$\dot{Q} = N + (P^t - Id)D.$$

In [19], Puhalskii obtains a LDP for the queue length process of a generalized Jackson network with a rate function that coincides with Theorem 2.1. Note that our model is slightly different here since we model the dynamic of a queue by a reflection mapping. Still in the case of Poisson processes for the inputs, both models correspond to the (exponential distribution) Jackson network. Recall that the rate function for a Poisson process of rate λ is given by (we keep the same notation as in 2.2),

$$\mathsf{I}^{\mathbf{C}}(\mathbf{x}) = \int_0^\infty \lambda \dot{\mathbf{x}}(t) \log \frac{\dot{\mathbf{x}}(t)}{\lambda} - \dot{\mathbf{x}}(t) + \lambda \, dt, \tag{2.6}$$

for absolutely continuous functions $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+)$. Hence if we replace (2.6) in the expression of $I^{\mathbf{Q}}_q$, we obtain the rate function for the large deviations of a Jackson network. In this specific case, the rate function has been obtained in different forms by Atar and Dupuis [3] and Igniatiouk-Robert [14] and some bounds have been computed by Majewski [18]. Compare to these results, our representation has the advantage of being quite intuitive, in the sense that each term is easy to interpret. If we interpret D, P, N as instantaneous departure, routing and exogenous arrival rates, then $N + (P^t - Id)D$ is just the vector of rates at which the queue lengths vary. Hence given a rate of change of $\mathbf{Q}$, the system behaves in such a way to minimize the instantaneous "costs" of departure, routing and exogenous arrival rates over all the rates that yield the desired $\dot{\mathbf{Q}}$.

From a methodological point of view, the argument of [19] is quite different from ours since the density condition (that we could compare to our Proposition 1.1) is verified on the rate function $\mathsf{I}^{\mathbf{Q}}_q$ (see condition (D) in [19]) whereas we are checking the density argument on the rate function of the inputs.

3. Extension of Ψ to piece-wise linear networks

In this section we consider processes that are continuous, i.e. in $\mathbb{C}(E)$, hence topological concepts refer to the compact uniform topology.

We first recall Proposition 3.2 of [16],

Proposition 3.1. *Given a $K \times K$ substochastic matrix P with $\rho(P) < 1$ and vectors $(\alpha, y) \in \mathbb{R}_+^{2K}$, the fixed point equation*

$$x^{(i)} \quad = \quad \alpha^{(i)} + \sum_{j=1}^{K} P^{(j,i)} \left(x^{(j)} \wedge y^{(j)} \right),$$

has a unique solution $x(y, P, \alpha)$. Moreover, $(y, \alpha) \mapsto x(y, P, \alpha)$ is a continuous non-decreasing function.

We first consider a linear network **Net** and show that the mapping Ψ (defined as the solution of the fixed-point Equation (2.2)) is well defined for such a network. By linear, we mean the following $\mathbf{N}^{(i)}(t) = N^{(i)} + \lambda^{(i)} t$, with $\lambda^{(i)} \geq 0$ and $N^{(i)} \in \mathbb{R}_+$, $\mathbf{S}^{(i)}(t) = \mu^{(i)} t$, with $\mu^{(i)} \geq 0$, and $\mathbf{P}^{(i,j)}(t) = P^{(i,j)} t$. We assume that $\rho(P) < 1$.

Lemma 3.1. *Under previous assumptions, the fixed point equation (0.1) has an unique solution $\mathbf{X}_f[\mu, P, N, \lambda](t) = x(\mu t, P, N + \lambda t)$, where $\mu = (\mu^{(i)})_i$, $N = (N^{(i)})_i$ and $\lambda = (\lambda^{(i)})_i$.*

Proof. Since μ, P, N, λ are fixed here, we omit to explicitly write the dependence in these variables. In this case, the fixed point equation (0.1) reduces to (see Remark 2.1)

$$\left\{ \begin{array}{lcl} \mathbf{A}^{(i)}(t) & = & N^{(i)} + \lambda^{(i)} t + \sum_{j=1}^{K} P^{(j,i)} \mathbf{D}^{(j)}(t), \\ \mathbf{D}^{(i)}(t) & = & \mu^{(i)} t \wedge \inf_{0 \leq s \leq t} \{ \mathbf{A}^{(i)}(s) + \mu^{(i)}(t-s) \}. \end{array} \right. \tag{3.1}$$

Thanks to Proposition 3.1, $\mathbf{X}_f(t) = x(\mu t, P, N + \lambda t)$ is the unique solution of the fixed point equation

$$\left\{ \begin{array}{lcl} \mathbf{A}^{(i)}(t) & = & N^{(i)} + \lambda^{(i)} t + \sum_{j=1}^{K} P^{(j,i)} \mathbf{D}^{(j)}(t), \\ \mathbf{D}^{(i)}(t) & = & \mathbf{A}^{(i)}(t) \wedge \mu^{(i)} t. \end{array} \right. \tag{3.2}$$

We prove now that $\mathbf{X}_f$ is the unique solution of the fixed point equation (3.1).

For simplicity, we denote the fixed point equation (3.1), resp. (3.2), by $\mathbf{A} = F(\mathbf{A})$, resp. by $\mathbf{A} = \tilde{F}(\mathbf{A})$. Note that these functions are non-decreasing, continuous and such that $F \leq \tilde{F}$.

From $\mathbf{0} \leq \mathbf{X}_f$, we get $\mathbf{0} \leq F(\mathbf{0}) \leq \tilde{F}(\mathbf{0}) \leq \tilde{F}(\mathbf{X}_f)$. Hence $F^n(\mathbf{0}) \nearrow \mathbf{L} \leq \mathbf{X}_f$ and $F(\mathbf{L}) = \mathbf{L}$. Moreover for any solution $\mathbf{Y}$ of the fixed point equation (3.1), we have $\mathbf{L} \leq \mathbf{Y} \leq \mathbf{X}_f$ because $\mathbf{Y} = F(\mathbf{Y}) \leq \tilde{F}(\mathbf{Y})$ and $\tilde{F}^n(\mathbf{Y}) \nearrow \mathbf{X}_f$.

Since $\mathbf{0}$ is a concave function, we have $F(\mathbf{0}) = \tilde{F}(\mathbf{0})$ and hence it is still a concave function. Hence we have $\tilde{F}^n(\mathbf{0}) = F^n(\mathbf{0})$ since the image by $\tilde{F}$ of a concave function is a concave function and $F = \tilde{F}$ on the subspace of concave functions. Hence we have $\mathbf{L} = \mathbf{X}_f$ which concludes the proof. ∎

In order, to extend Ψ to piece-wise linear networks, we proceed step by step on each interval where the driving functions $\mathbf{S}, \mathbf{P}, \mathbf{N}$ are linear. The following lemma allows to glue the constructed solution on each adjacent interval.

Lemma 3.2. *Let* $\mathbf{A}, \mathbf{S} \in \mathbb{D}(\mathbb{R}_+) \times \mathbb{D}_0(\mathbb{R}_+)$ *and* $\mathbf{D} = \Phi(\mathbf{A}, \mathbf{S})$*. Define* $\tilde{\mathbf{A}}, \tilde{\mathbf{S}} \in \mathbb{D}(\mathbb{R}_+) \times \mathbb{D}_0(\mathbb{R}_+)$ *as follows*

$$\begin{aligned} \tilde{\mathbf{A}}(t) &:= \mathbf{A}(t+u) - \mathbf{D}(u), \\ \tilde{\mathbf{S}}(t) &:= \mathbf{S}(t+u) - \mathbf{S}(u). \end{aligned}$$

Let $\tilde{\mathbf{D}} = \Phi(\tilde{\mathbf{A}}, \tilde{\mathbf{S}})$*, then we have*

$$\tilde{\mathbf{D}}(t) = \mathbf{D}(t+u) - \mathbf{D}(u).$$

Proof. We show that for $\mathbf{D} = \Phi(\mathbf{A}, \mathbf{S})$, we have

$$\mathbf{D}(t+u) - \mathbf{D}(u) = \inf_{u \le s \le t+u} \left\{ \mathbf{S}(t+u) - \mathbf{S}(s) + \mathbf{A}(s) - \mathbf{D}(u) \right\} \wedge \left\{ \mathbf{S}(t+u) - \mathbf{S}(u) \right\},$$

from which the lemma follows.

We write

$$\begin{aligned} \mathbf{D}(t+u) - \mathbf{D}(u) &= \inf_{0 \le s \le u} \left\{ \mathbf{S}(t+u) - \mathbf{S}(s) + \mathbf{A}(s) - \mathbf{D}(u) \right\} \\ &\quad \wedge \inf_{u \le s \le t+u} \left\{ \mathbf{S}(t+u) - \mathbf{S}(s) + \mathbf{A}(s) - \mathbf{D}(u) \right\} \wedge \left\{ \mathbf{S}(t+u) - \mathbf{D}(u) \right\}, \end{aligned}$$

Since $\mathbf{D}(u) \le \mathbf{S}(u)$, we have to prove that

$$\mathbf{S}(t+u) - \mathbf{S}(u) \ge \inf_{0 \le s \le u} \left\{ \mathbf{S}(t+u) - \mathbf{S}(s) + \mathbf{A}(s) - \mathbf{D}(u) \right\} \wedge \left\{ \mathbf{S}(t+u) - \mathbf{D}(u) \right\}.$$

This will follow from,

$$\begin{aligned} \inf_{0 \le s \le u} \left\{ \mathbf{S}(t+u) - \mathbf{S}(s) + \mathbf{A}(s) - \mathbf{D}(u) \right\} &= \mathbf{S}(t+u) - \mathbf{S}(u) + \inf_{0 \le s \le u} \left\{ \mathbf{S}(u) - \mathbf{S}(s) + \mathbf{A}(s) \right\} - \mathbf{D}(u) \\ &\le \mathbf{S}(t+u) - \mathbf{S}(u). \end{aligned}$$

□

We consider now piece-wise linear networks: the functions $u \mapsto \mathbf{N}^{(i)}(u), u \mapsto \mathbf{S}^{(i)}(u)$ and $u \mapsto \mathbf{P}^{(i,j)}(u)$ are continuous piece-wise linear functions such that $\mathbf{N}^{(i)}(0) \in \mathbb{R}_+$ and $\mathbf{S}^{(i)}(0) = \mathbf{P}^{(i,j)}(0) = 0$ and $\rho(\dot{\mathbf{P}}(t)) < 1$ for all $t \ge 0$.

Proposition 3.1. *For a piece-wise linear network, there exists an unique solution of the fixed point equation (0.1). We still denote by* Ψ *the mapping that to any piece-wise linear network* **Net** *associates the corresponding couple* $(\mathbf{A}, \mathbf{D})$*.*

Proof. The existence is a direct consequence of monotonicity properties and continuity of the maps Γ and Φ. We define the sequence of processes $\{\mathbf{A}[k], \mathbf{D}[k]\}_{k \ge 0}$ with the recurrence equation:

$$\begin{cases} \mathbf{A}[k+1] = \Gamma(\mathbf{D}[k], \mathbf{Net}), \\ \mathbf{D}[k+1] = \Phi(\mathbf{A}[k+1], \mathbf{Net}), \end{cases}$$

and with initial condition $\mathbf{D}[0] = \mathbf{0}$. By the monotonicity properties of Φ and Γ, we have

$$\begin{aligned} \mathbf{0} \le \mathbf{A}[1] &\Rightarrow \Phi(\mathbf{0}, \mathbf{Net}) = \mathbf{0} = \mathbf{D}[0] \le \Phi(\mathbf{A}[1], \mathbf{Net}) = \mathbf{D}[1] \\ &\Rightarrow \Gamma(\mathbf{D}[0], \mathbf{Net}) = \mathbf{A}[1] \le \Gamma(\mathbf{D}[1], \mathbf{Net}) = \mathbf{A}[2], \end{aligned}$$

and the sequence $\{\mathbf{A}[k], \mathbf{D}[k]\}_{k \ge 0}$ is increasing. Note that $\mathbf{D}[k] \le \mathbf{S}$ and hence the following limits are well defined

$$\lim_{k \to \infty} \mathbf{A}[k] = \mathbf{A} \quad \text{and,} \quad \lim_{k \to \infty} \mathbf{D}[k] = \mathbf{D}.$$

Since Γ and Φ are continuous, $(\mathbf{A}, \mathbf{D})$ is a solution of the fixed point equation (0.1).

We now prove uniqueness. First recall that we call α, a partition of $\mathbb{R}_+$, any increasing sequence of points $\alpha = \{a_n\}_n$ with $a_0 = 0$ and $a_n \to \infty$. For two partitions $\alpha = \{a_n\}_n$ and $\beta = \{b_n\}_n$, we say that $\gamma = \{g_n\}_n$ is the union of α and β if γ is a partition such that for all n there exists m such that either $g_n = a_m$ or $g_n = b_m$.

Let $\tau = \{t_n\}_n$ be the union of the partitions associated with each function $\mathbf{S}, \mathbf{P}, \mathbf{N}$. We define for $x \in \mathbb{R}_+$, $d(x,\tau) = \min_n\{t_n - x,\ t_n > x\} > 0$.

Assume that we are given two solutions of the fixed point equation (0.1): $(\mathbf{A}_1, \mathbf{D}_1)$ and $(\mathbf{A}_2, \mathbf{D}_2)$. First note that thanks to Lemmas 5.1 and 5.2, any solution of (0.1) is absolutely continuous. Let $z = \inf\{t, \mathbf{A}_1(t) \neq \mathbf{A}_2(t)\}$, in particular, we have $\mathbf{A}_1(t) = \mathbf{A}_2(t)$ and $\mathbf{D}_1(t) = \mathbf{D}_2(t)$ for all $t \le z$.

Define $u = \min_i d(\mathbf{D}^{(i)}_\bullet(z), \tau) \wedge d(z,\tau) > 0$, where the notation $\bullet$ can be replaced either by $_1$ or by $_2$. We have that for $t \in [0,u]$,

$$\begin{aligned}
\tilde{\mathbf{S}}^{(i)}(t) &:= \mathbf{S}^{(i)}(z+t) - \mathbf{S}^{(i)}(z) = t\mu^{(i)},\\
\tilde{\mathbf{P}}^{(i,j)}(t) &:= \mathbf{P}^{(i,j)}(\mathbf{D}^{(i)}_\bullet(z)+t) - \mathbf{P}^{(i,j)}(\mathbf{D}^{(i)}_\bullet(z)) = tP^{(i,j)},\\
\tilde{\mathbf{N}}^{(i)} &:= \mathbf{N}^{(i)}(z+t) - \mathbf{N}^{(i)}(z) + \mathbf{A}^{(i)}_\bullet(z) - \mathbf{D}^{(i)}_\bullet(z) = t\lambda^{(i)} + \mathbf{A}^{(i)}_\bullet(z) - \mathbf{D}^{(i)}_\bullet(z),
\end{aligned}$$

Let $\tilde{\mathbf{A}}(t) = \mathbf{X}_f[\mu, P, \mathbf{A}_\bullet(z) - \mathbf{D}_\bullet(z), \lambda](t)$ be the unique solution associated to the infinite horizon linear network defined above. The associated departure process is $\tilde{\mathbf{D}}(t) = \tilde{\mathbf{A}}(t) \wedge \mu t$. Let $v = \inf\{t, \inf_i \tilde{\mathbf{D}}^{(i)}(t) = u\}$, in particular since $\tilde{\mathbf{D}}^{(i)}(t) \le \mu^{(i)} t$, we have $v > 0$. In view of Lemma 3.2, we have for $t \in (0,v)$,

$$\mathbf{A}_\bullet(t+z) = \tilde{\mathbf{A}}(t) + \mathbf{D}(z), \quad \mathbf{D}_\bullet(t+z) = \tilde{\mathbf{D}}(t) + \mathbf{D}(z)$$

this contradicts the fact that $z < \infty$ and concludes the proof. $\square$

Let $\mathcal{E} \subset \mathbb{D}_0(\mathbb{R}_+^K) \times \mathbb{D}_0(\mathbb{M}^K) \times \mathbb{D}(\mathbb{R}_+^K)$ as defined at the beginning of Section 2.2 and $\mathcal{F} = \mathbb{D}(\mathbb{R}_+^K) \times \mathbb{D}_0(\mathbb{R}_+^K)$.

For $\mathbf{Net} \in \mathcal{E}$ and $(\mathbf{A}, \mathbf{D}) \in \mathcal{F}$, we define the function

$$G(\mathbf{Net}, \mathbf{A}, \mathbf{D}) = \|(\mathbf{A} - \Gamma(\mathbf{D}, \mathbf{Net}), \mathbf{D} - \Phi(\mathbf{A}, \mathbf{Net}))\|.$$

The function G is continuous and such that

$$G(\mathbf{Net}, \mathbf{A}, \mathbf{D}) = 0 \Leftrightarrow \left\{ \begin{array}{lcl} \mathbf{A} &=& \Gamma(\mathbf{D}, \mathbf{Net}),\\ \mathbf{D} &=& \Phi(\mathbf{A}, \mathbf{Net}). \end{array}\right.$$

Let $\mathcal{D}_{\mathbf{Net}}$ be the subspace of $\mathcal{E}$ of piecewise linear networks: namely $\mathbf{Net} = (\mathbf{S}, \mathbf{P}, \mathbf{N}) \in \mathcal{D}_{\mathbf{Net}}$ if the functions $u \mapsto \mathbf{N}^{(i)}(u), u \mapsto \mathbf{S}^{(i)}(u)$ and $u \mapsto \mathbf{P}^{(i,j)}(u)$ are piecewise linear non-decreasing functions such that $\rho(\dot{\mathbf{P}}(t)) < 1$ for all $t \ge 0$ and $\mathbf{N}^{(i)} = \mathbf{0}$ for $i \notin \mathcal{S}$. We denote $\dot{\mathbf{Net}} = (\dot{\mathbf{S}}, \dot{\mathbf{P}}, \dot{\mathbf{N}})$.

We proved that

$$\forall \mathbf{Net} \in \mathcal{D}_{\mathbf{Net}}, \quad G(\mathbf{Net}, \mathbf{A}, \mathbf{D}) = 0 \Leftrightarrow (\mathbf{A}, \mathbf{D}) = \Psi(\mathbf{Net}),$$

where Ψ has been explicitly defined above. We are exactly in the framework of Section 1. In the next section we construct the mapping $\mathcal{S} : \mathcal{E} \times \mathcal{F} \to \mathcal{D}_{\mathbf{Net}}^{\mathbb{N}}$.

4. Sample path large deviations

In order to simplify the notations, we assume that $\mathbf{N}_n(0) = 0$ for all n. This condition can be weakened to the standard condition:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\mathbf{N}_n(0) > \epsilon) = 0,$$

for all $\epsilon > 0$. In this case, we have $\mathsf{I}^0(x) = \infty$ for all $x \neq 0$ and $\mathsf{I}^0(0) = 0$.

It is possible to deal with the case where the initial condition satisfies a LDP as assumed in Theorem 2.1 by using a standard conditioning argument (as done in [21] for example).

4.1. **Construction of the approximating sequence.** This section is devoted to the proof of the following proposition:

Proposition 4.1. *We consider* $\mathbf{Net} = (\mathbf{S}, \mathbf{P}, \mathbf{N}) \in \mathcal{E}$ *such that* $I^{\mathbf{Net}}(\mathbf{Net}) < \infty$ *and such that there exists* $(\mathbf{A}, \mathbf{D}) \in \mathcal{F}$ *that satisfies the fixed point equation (0.1) given by,*

$$\left\{ \begin{array}{lcl} \mathbf{A} & = & \Gamma(\mathbf{D}, \mathbf{Net}), \\ \mathbf{D} & = & \Phi(\mathbf{A}, \mathbf{Net}). \end{array} \right.$$

There exists a sequence $\{\mathbf{Net}_n\}_n = \mathcal{S}(\mathbf{Net}, \mathbf{A}, \mathbf{D})$ *such that*

$$\mathbf{Net}_n \in \mathcal{D}_{\mathbf{Net}} \quad \text{for all } n; \tag{4.1}$$

$$\mathbf{Net}_n \to \mathbf{Net}; \tag{4.2}$$

$$\Psi(\mathbf{Net}_n) \to (\mathbf{A}, \mathbf{D}); \tag{4.3}$$

$$\mathsf{I}^{\mathbf{Net}}(\mathbf{Net}_n) \to \mathsf{I}^{\mathbf{Net}}(\mathbf{Net}). \tag{4.4}$$

First note that since $\mathsf{I}^{\mathbf{Net}}(\mathbf{Net}) < \infty$, each process $\mathbf{S}, \mathbf{P}, \mathbf{N}$ is absolutely continuous and $\dot{\mathbf{Net}}$ is well-defined. Moreover thanks to Lemma 5.3, the processes $\mathbf{A}$ and $\mathbf{D}$ are absolutely continuous too.

The idea to construct the sequence $\{\mathbf{Net}_n\}_n$ is to consider the piecewise approximation of the fixed point equation (0.1). First consider the routing equation $\mathbf{A} = \Gamma(\mathbf{D}, \mathbf{Net})$ for times t such that $nt \in \mathbb{N}$,

$$\underbrace{\mathbf{A}^{(i)}(t+1/n) - \mathbf{A}^{(i)}(t)}_{\Delta_n^{(i)}(\mathbf{A})(t)} = \underbrace{\mathbf{N}^{(i)}(t+1/n) - \mathbf{N}^{(i)}(t)}_{\Delta_n^{(i)}(\mathbf{N})(t)} + \sum_{j=1}^{K} \dot{\tilde{\mathbf{P}}}_n^{(j,i)}(\mathbf{D}^{(j)}(t+)) \underbrace{(\mathbf{D}^{(j)}(t+1/n) - \mathbf{D}^{(j)}(t))}_{\Delta_n^{(j)}(\mathbf{D})(t)},$$

where we define the piece-wise linear process $\tilde{\mathbf{P}}_n^{(j,i)}(t)$ as follows, for $s \in (\mathbf{D}^{(j)}(t), \mathbf{D}^{(j)}(t+1/n))$,

$$\dot{\tilde{\mathbf{P}}}_n^{(j,i)}(s) := \frac{\mathbf{P}^{(j,i)}(\mathbf{D}^{(j)}(t+1/n)) - \mathbf{P}^{(j,i)}(\mathbf{D}^{(j)}(t))}{\mathbf{D}^{(j)}(t+1/n) - \mathbf{D}^{(j)}(t)},$$

if $\mathbf{D}^{(j)}(t+1/n) \neq \mathbf{D}^{(j)}(t)$, and we take $\dot{\tilde{\mathbf{P}}}_n^{(j,i)}(\mathbf{D}^{(j)}(t)) = 0$ otherwise. In other words, we have

$$\begin{array}{rcl} \tilde{\mathbf{P}}_n^{(j,i)}(\mathbf{D}^{(j)}(t+1/n)) - \tilde{\mathbf{P}}_n^{(j,i)}(\mathbf{D}^{(j)}(t)) & = & \dot{\tilde{\mathbf{P}}}_n^{(j,i)}(\mathbf{D}^{(j)}(t+))(\mathbf{D}^{(j)}(t+1/n) - \mathbf{D}^{(j)}(t)) \\ & = & \mathbf{P}^{(j,i)}(\mathbf{D}^{(j)}(t+1/n)) - \mathbf{P}^{(j,i)}(\mathbf{D}^{(j)}(t)) \end{array}$$

Note that $\{\dot{\tilde{\mathbf{P}}}_n^{(j,i)}(t)\}_{i,j} \in \mathbb{M}^K$ since we have by the definition of $\mathcal{E}$,

$$\sum_i \mathbf{P}^{(j,i)}(\mathbf{D}^{(j)}(t+1/n)) - \mathbf{P}^{(j,i)}(\mathbf{D}^{(j)}(t)) \le \mathbf{D}^{(j)}(t+1/n) - \mathbf{D}^{(j)}(t),$$

but the matrix $(\dot{\tilde{\mathbf{P}}}_n^{(j,i)}(\mathbf{D}^{(j)}(t+)))_{i,j}$ may not be of spectral radius less than 1.

To circumvent this difficulty, we modify slightly the processes as follows, (the variables η, ϵ_n, δ will be made precise latter)

$$\begin{aligned}\Delta_n^{(i)}(\mathbf{A}) + \frac{\eta^{(i)}}{n} &= \Delta_n^{(i)}(\mathbf{N}) + \frac{\delta^{(i)}}{n} \\ &+ \sum_{j=1}^K \left((1-\epsilon_n^{(j)})\dot{\tilde{\mathbf{P}}}_n^{(j,i)} + \epsilon_n^{(j)} R^{(j,i)}\right)\left(\Delta_n^{(j)}(\mathbf{D}) + \frac{\eta^{(j)}}{n}\right),\end{aligned} \tag{4.5}$$

where we omit to write the time t and use the simplified notation $\dot{\tilde{\mathbf{P}}}_n^{(j,i)} = \dot{\tilde{\mathbf{P}}}_n^{(j,i)}(\mathbf{D}^{(j)}(t+))$.

We have to find η, ϵ_n, δ such that (4.5) holds with $\eta^{(i)}, \epsilon_n^{(i)}, \delta^{(i)}$ non-negative and $\delta^{(i)} = 0$ for $i \notin \mathcal{S}$. These constraints are satisfied by the following choice: first take δ such that $\delta^{(i)} > 0$ for all $i \in \mathcal{S}$ and $\delta^{(i)} = 0$ for $i \notin \mathcal{S}$. Let $\eta(\delta) = \eta$ be the unique solution in $\mathbb{R}_+^K$ of the following equation (recall that $\rho(R) < 1$),

$$\eta^{(i)} = \delta^{(i)} + \sum_{j=1}^K \eta^{(j)} R^{(j,i)}.$$

Note that $\eta^{(i)} > 0$ for all i thanks to (2.4). Finally let define $\epsilon_n(\delta) = \epsilon_n$ as follows $\epsilon_n^{(i)} = \frac{\eta^{(i)}}{n\Delta_n^{(i)}(\mathbf{D})+\eta^{(i)}} \in (0,1]$ (note that $\epsilon_n^{(i)} = 1$ if and only if $\Delta_n^{(i)}(\mathbf{D}) = 0$).

It is easy to see that (4.5) holds since we have

$$(1-\epsilon_n^{(j)})\left(\Delta_n^{(j)}(\mathbf{D}) + \frac{\eta^{(j)}}{n}\right) = \Delta_n^{(j)}(\mathbf{D}), \quad \text{or,} \quad \epsilon_n^{(j)}\left(\Delta_n^{(j)}(\mathbf{D}) + \frac{\eta^{(j)}}{n}\right) = \frac{\eta^{(j)}}{n},$$

which imply respectively that

$$\begin{aligned}\Delta_n^{(i)}(\mathbf{A}) &= \Delta_n^{(i)}(\mathbf{N}) + \sum_{j=1}^K (1-\epsilon_n^{(j)})\dot{\tilde{\mathbf{P}}}_n^{(j,i)}\left(\Delta_n^{(j)}(\mathbf{D}) + \frac{\eta^{(j)}}{n}\right) \quad \text{and,} \\ \frac{\eta^{(i)}}{n} &= \frac{\delta^{(i)}}{n} + \sum_{j=1}^K \epsilon_n^{(j)} R^{(j,i)}\left(\Delta_n^{(j)}(\mathbf{D}) + \frac{\eta^{(j)}}{n}\right),\end{aligned}$$

and summing these two equalities gives (4.5).

For δ fixed, we define for $s \in (\mathbf{D}^{(j)}(t) + t\eta(\delta), \mathbf{D}^{(j)}(t+1/n) + (t+1/n)\eta(\delta))$,

$$\dot{\mathbf{P}}_{n,\delta}^{(j,i)}(s) = (1-\epsilon_n^{(j)})\dot{\tilde{\mathbf{P}}}_n^{(j,i)}(\mathbf{D}^{(j)}(t+)) + \epsilon_n^{(j)} R^{(j,i)},$$

where $\epsilon_n(\delta)$ is defined as above. In view of Lemma 5.4, the matrix $\dot{\mathbf{P}}_{n,\delta}^{(j,i)}(s)$ is of spectral radius less than one since $\epsilon_n^{(j)} > 0$ for all j. Then as a direct consequence of (4.5), we have for $nt \in \mathbb{N}$,

$$\mathbf{A}^{(i)}(t) + t\eta(\delta) = \mathbf{N}^{(i)}(t) + t\delta + \sum_{j=1}^K \mathbf{P}_{n,\delta}^{(j,i)}(\mathbf{D}^{(j)}(t) + t\eta(\delta)). \tag{4.6}$$

If $\mathbf{N}_{n,\delta}$ is the polygonal approximation of $t \to \mathbf{N}(t) + t\delta$ with step $1/n$, we have clearly $\dot{\mathbf{N}}_{n,\delta} \to \dot{\mathbf{N}} + \delta$ as n tends to infinity. Similarly, we have as n tends to infinity,

$$\dot{\mathbf{P}}^{(j,i)}_{n,\delta}(\mathbf{D}^{(j)}(t) + t\eta(\delta)) \to \begin{cases} (1-\epsilon^{(j)}(t))\dot{\mathbf{P}}^{(j,i)}(\mathbf{D}^{(j)}(t))\frac{(\dot{\mathbf{D}}^{(j)}(t)+\eta(\delta))}{\dot{\mathbf{D}}^{(j)}(t)} & \\ \qquad +\epsilon^{(j)}(t)R^{(j,i)}(\dot{\mathbf{D}}^{(j)}(t)+\eta(\delta)) & \text{if } \dot{\mathbf{D}}^{(j)}(t) > 0, \\ R^{(j,i)}\eta(\delta) & \text{otherwise,} \end{cases}$$

where $\epsilon^{(j)}(t) = \eta^{(j)}(\delta)/(\eta^{(j)}(\delta) + \dot{\mathbf{D}}^{(j)}(t)) < 1$. Hence when n tends to infinity and δ tends to zero, we have $\dot{\mathbf{N}_{n,\delta}} \to \dot{\mathbf{N}}$ and $\dot{\mathbf{P}}^{(j,i)}_{n,\delta} \to \dot{\mathbf{P}}^{(j,i)}$.

We consider now the queueing equation $\mathbf{D} = \Phi(\mathbf{A}, \mathbf{S})$ and construct the approximating sequence for $\mathbf{S}$.

We begin with a first general lemma: given three processes $\mathbf{A} \le \mathbf{D}$ and $\mathbf{S}$, we construct a piecewise linear function $\mathbf{S}_n$ (with step $1/n$) as follows (with $nt \in \mathbb{N}$):

- if $\mathbf{A}(t) = \mathbf{D}(t)$ and $\mathbf{A}(t+1/n) = \mathbf{D}(t+1/n)$, then $\mathbf{S}_n(t+1/n) - \mathbf{S}_n(t) = \mathbf{S}(t+1/n) - \mathbf{S}(t)$;
- otherwise, $\mathbf{S}_n(t+1/n) - \mathbf{S}_n(t) = \mathbf{D}(t+1/n) - \mathbf{D}(t)$.

We will denote this construction by $\mathbf{S}_n = \Upsilon_n(\mathbf{A}, \mathbf{D}, \mathbf{S})$.

Lemma 4.1. *Let* $(\mathbf{A}, \mathbf{D}, \mathbf{S})$ *be absolutely continuous functions of* $\mathbb{C}(\mathbb{R}_+^K) \times \mathbb{C}_0(\mathbb{R}_+^K) \times \mathbb{C}_0(\mathbb{R}_+^K)$ *such that* $\Phi(\mathbf{A}, \mathbf{S}) = \mathbf{D}$. *We denote* $\mathbf{S}_n = \Upsilon_n(\mathbf{A}, \mathbf{D}, \mathbf{S})$. *We have* $\mathbf{D}_n = \Phi(\mathbf{A}_n, \mathbf{S}_n)$ *where* $(\mathbf{A}_n, \mathbf{D}_n)$ *is the polygonal approximation of* $(\mathbf{A}, \mathbf{D})$ *with step* $1/n$ *and we have the following convergence as* n *tends to infinity:* $\mathbf{S}_n \to \mathbf{S}$, $\dot{\mathbf{S}}_n \to \dot{\mathbf{S}}$ *and* $\int_0^\infty \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_n(t))dt \to \int_0^\infty \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t))dt$ *.*

Proof. We denote $\tilde{\mathbf{D}}_n = \Phi(\mathbf{A}_n, \mathbf{S}_n)$. From the proof of Lemma 3.2, we have

$$\tilde{\mathbf{D}}_n(t+1/n) - \tilde{\mathbf{D}}_n(t) = \inf_{t \le s \le t+1/n} \left\{ \mathbf{S}_n(t+1/n) - \mathbf{S}_n(s) + \mathbf{A}_n(s) - \tilde{\mathbf{D}}_n(t) \right\} \wedge \left\{ \mathbf{S}_n(t+1/n) - \mathbf{S}_n(t) \right\},$$

since all the functions are linear on the interval $(t, t+1/n)$, we have (with $nt \in \mathbb{N}$),

$$\tilde{\mathbf{D}}_n(t+1/n) - \tilde{\mathbf{D}}_n(t) = \left\{ \mathbf{A}_n(t+1/n) - \tilde{\mathbf{D}}_n(t) \right\} \wedge \left\{ \mathbf{S}_n(t+1/n) - \mathbf{S}_n(t) \right\}.$$

If $\tilde{\mathbf{D}}_n(t) = \mathbf{D}_n(t)$, then we have clearly $\tilde{\mathbf{D}}_n(t+1/n) = \mathbf{D}_n(t+1/n)$ since

- if $\mathbf{A}_n(t) = \mathbf{D}_n(t)$ and $\mathbf{A}_n(t+1/n) = \mathbf{D}_n(t+1/n)$, then we have $\mathbf{S}(t+1/n) - \mathbf{S}(t) \ge \mathbf{D}_n(t+1/n) - \mathbf{D}_n(t) = \mathbf{A}_n(t+1/n) - \tilde{\mathbf{D}}_n(t)$ see (5.1) for the inequality;
- otherwise, $\mathbf{S}_n(t+1/n) - \mathbf{S}_n(t) = \mathbf{D}_n(t+1/n) - \mathbf{D}_n(t)$ by definition and $\mathbf{A}_n(t+1/n) \ge \mathbf{D}_n(t+1/n)$.

This proves the first part of the lemma. Moreover it follows directly form the definition of Υ that $\mathbf{S}_n\left(t+\frac{1}{n}\right) - \mathbf{S}_n(t) \le \mathbf{S}\left(t+\frac{1}{n}\right) - \mathbf{S}(t)$, hence we have for all t, $\limsup_{n\to\infty} \mathbf{S}_n(t) \le \mathbf{S}(t)$ by a continuity argument. The fact that $\mathbf{S}_n \to \mathbf{S}$ follows directly from Fatou's Lemma and the fact that $\dot{\mathbf{S}}_n \to \dot{\mathbf{S}}$. We now prove this last fact, let $C = \{t,\ \mathbf{A}(t) = \mathbf{D}(t)\}$. C is a closed set and according to Lemma 5.3, we have for all $t \in C^c$ (the complementary set of C), $\dot{\mathbf{S}}(t) = \dot{\mathbf{D}}(t)$. For such $t \in C^c$, we have for $\epsilon > 0$ sufficiently small and for sufficiently large n, $\mathbf{A}_n(u) \ne \mathbf{D}_n(u)$ for all $|u-t| \le \epsilon$. Hence we have $\dot{\mathbf{S}}_n(t) = \dot{\mathbf{D}}_n(t) \to \dot{\mathbf{D}}(t)$. Now for $t \in C^o$ in the interior of C, we have clearly $\dot{\mathbf{S}}_n(t) \to \dot{\mathbf{S}}(t)$. Hence we have $\dot{\mathbf{S}}_n(t) \to \dot{\mathbf{S}}(t)$ for $t \in C^o \cup C^c$.

We prove the last statement of the lemma. Since any open set of $\mathbb{R}$ is a countable union of disjoint intervals,

$$\begin{aligned}\int_{C^c} \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_n(t))dt = \int_{C^c} \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{D}}_n(t))dt &\leq \int_{C^c} \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{D}}(t))dt, \quad \text{by Jensen's inequality}\\ &= \int_{C^c} \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t))dt,\end{aligned}$$

and also directly still by Jensen's inequality $\int_{C^o} \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_n(t))dt \leq \int_{C^o} \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t))dt$. The convergence then follows from

$$\begin{aligned}\liminf_{n\to\infty} \int_0^\infty \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_n(t))dt &\geq \int_0^\infty \liminf_{n\to\infty} \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_n(t))dt\\ &\geq \int_0^\infty \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t))dt,\end{aligned}$$

where the first inequality is due to Fatou's Lemma and the second one to the lower semicontinuity of $\mathsf{I}^{\mathbf{S}}$. □

We define the sequence $\mathbf{Net}_{n,\delta} = (\mathbf{S}_{n,\delta}, \mathbf{P}_{n,\delta}, \mathbf{N}_{n,\delta})$ where $\mathbf{S}_{n,\delta}(t) = \Upsilon_n(\mathbf{A}(t)+\eta t, \mathbf{D}(t)+\eta t, \mathbf{S}(t)+\eta t)$. Note that we have $\mathbf{D}(t)+\eta t = \Phi(\mathbf{A}(t)+\eta t, \mathbf{S}(t)+\eta t)$, hence Lemma 4.1 applies, in particular, we have $\dot{\mathbf{S}}_{n,\delta}(t) \to \dot{\mathbf{S}}(t)+\eta(\delta)$ as n tends to infinity.

We have $\mathbf{Net}_{n,\delta} \in \mathcal{D}_{\mathbf{Net}}$ by construction and the sequence $\{\mathbf{Net}_{n,\delta_n}\}_n$ satisfies (4.2) for some $\delta_n \to 0$. Moreover, we have thanks to (4.6) and Lemma 4.1,

$$\left\{\begin{array}{lcl}\mathbf{A}_{n,\delta} &=& \Gamma(\mathbf{D}_{n,\delta}, \mathbf{Net}_{n,\delta}),\\ \mathbf{D}_{n,\delta} &=& \Phi(\mathbf{A}_{n,\delta}, \mathbf{Net}_{n,\delta}),\end{array}\right. \Leftrightarrow (\mathbf{A}_{n,\delta}, \mathbf{D}_{n,\delta}) = \Psi(\mathbf{Net}_{n,\delta}),$$

where $\mathbf{A}_{n,\delta}$ and $\mathbf{D}_{n,\delta}$ are the polygonal approximation of $\mathbf{A}(t)+\eta t$ and $\mathbf{D}(t)+\eta t$ with step $1/n$ and Ψ has been defined in Section 3.

For $n\to\infty$ and $\delta\to 0$, we have $(\mathbf{A}_{n,\delta}, \mathbf{D}_{n,\delta}) \to (\mathbf{A}, \mathbf{D})$, hence we have $\Psi(\mathbf{Net}_{n,\delta}) \to (\mathbf{A}, \mathbf{D})$, i.e. the sequence $\{\mathbf{Net}_{n,\delta_n}\}_n$ satisfies (4.3).

We now show that (4.4) is also satisfied. We fix $T>0$ and prove first that we have, for δ sufficiently small,

$$\begin{aligned}&\int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t)) + \tilde{\mathsf{D}}(\dot{\mathbf{P}}(t)\|R) + \mathsf{I}^{\mathbf{N}}(\dot{\mathbf{N}}(t))dt - er(\delta)T \qquad (4.7)\\ &\leq \liminf_{n\to\infty} \int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_{n,\delta}(t)) + \tilde{\mathsf{D}}(\dot{\mathbf{P}}_{n,\delta}(t)\|R) + \mathsf{I}^{\mathbf{N}}(\dot{\mathbf{N}}_{n,\delta}(t))dt\\ &\leq \limsup_{n\to\infty} \int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_{n,\delta}(t)) + \tilde{\mathsf{D}}(\dot{\mathbf{P}}_{n,\delta}(t)\|R) + \mathsf{I}^{\mathbf{N}}(\dot{\mathbf{N}}_{n,\delta}(t))dt\\ &\leq \int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t)) + \tilde{\mathsf{D}}(\dot{\mathbf{P}}(t)\|R) + \mathsf{I}^{\mathbf{N}}(\dot{\mathbf{N}}(t))dt + er(\delta)T, \qquad (4.8)\end{aligned}$$

where $er(\delta)$ tends to zero as δ tends to zero, from which (4.4) follows by monotonicity.

We first deal with the case of the sequence of processes $\{\mathbf{S}_{n,\delta}\}_n$ (we can restrict ourselves to the one dimensional case). We denote $\mathbf{S}_\delta(t) = \mathbf{S}(t)+\eta(\delta)t$.

We define $\varsigma = \text{esssup}\{\dot{\mathbf{S}}(t),\ t\leq T\} = \inf\{u,\ Leb[t\leq T,\ \dot{\mathbf{S}}(t)>u]=0\}$, where Leb is for the Lebesgue measure. Since $\int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t))dt<\infty$, ς belongs to the domain of $\mathsf{I}^{\mathbf{S}}$ which is open on the right. Hence we can find $\epsilon>0$ such that $\varsigma+\epsilon$ still belongs to this domain and take δ such that

$\eta(\delta) < \epsilon$. Moreover, since $\mathsf{I}^{\mathbf{S}}$ is convex, it is uniformly continuous on $[0, \varsigma + \epsilon]$. Hence, we can assume that we have $\beta(\alpha) \to 0$ as $\alpha \to 0$ such that,

$$\forall x, y \in [0, \varsigma + \epsilon],\ |x - y| < \alpha \Rightarrow |\mathsf{I}^{\mathbf{S}}(x) - \mathsf{I}^{\mathbf{S}}(y)| \leq \beta(\alpha).$$

From Lemma 4.1, we have

$$\begin{aligned}\int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t))dt - \beta(\eta(\delta))T \leq &\\ \lim_{n\to\infty} \int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_{n,\delta}(t))dt &= \int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}_{\delta}(t))dt \\ &\leq \int_0^T \mathsf{I}^{\mathbf{S}}(\dot{\mathbf{S}}(t))dt + \beta(\eta(\delta))T. \end{aligned} \tag{4.9}$$

Hence we proved (4.7) and (4.8) for $\mathsf{I}^{\mathbf{S}}$.

The same kind of arguments can be repeated for $\mathbf{N}_{n,\delta}$ which is just the polygonal approximation of $t \mapsto \mathbf{N}(t) + \delta t$. Note that $\{\mathbf{Net}_{n,\delta}\}_n \in \mathcal{D}^{\mathbb{N}}_{\mathbf{Net}}$ implies that $\mathbf{N}^{(i)}_{n,\delta}(t) = 0$ for all $i \notin \mathcal{S}$. For $i \in \mathcal{S}$, we can use the fact that the domain of $\mathsf{I}^{\mathbf{N}^{(i)}}$ is open as previously. In the case of $\mathbf{P}_{n,\delta}$, we can not use the argument on the openness of the domain, but we have $\tilde{\mathsf{D}}(R^{(i)} \| R^{(i)}) = 0$ and then the convexity of $\tilde{\mathsf{D}}$ directly implies that $\tilde{\mathsf{D}}(\dot{\mathbf{P}}^{(i)}_{n,\delta} \| R^{(i)}) \leq \tilde{\mathsf{D}}(\dot{\mathbf{P}}^{(i)} \| R^{(i)})$, from which we derive an equivalent of (4.9).

4.2. **Exponential tightness.** We first recall some definitions. A sequence of random variables $\{X_n\}_n \in (\mathbb{R}^K)^{\mathbb{N}}$ is exponentially tight if

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\|X_n\| > M) = -\infty.$$

For $\delta > 0$ and $T > 0$, define the modulus of continuity in $\mathbb{D}(E)$ by

$$w'(\mathbf{X}, \delta, T) := \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} d(\mathbf{X}(s), \mathbf{X}(t)),$$

where the infimum is over $\{t_i\}$ satisfying

$$0 = t_0 < t_1 < \cdots < t_{m-1} < T \leq t_m$$

and $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \delta$.

Theorem 4.1 of [11] tells us: let $\mathcal{T}_0$ be a dense subset of $\mathbb{R}_+$. Suppose that for each $t \in \mathcal{T}_0$, $\{\mathbf{X}_n(t)\}_n$ is exponentially tight. Then $\{\mathbf{X}_n\}_n$ is exponentially tight in $\mathbb{D}(E)$ if and only if for each $\epsilon > 0$ and $T > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(w'(\mathbf{X}_n, \delta, T) > \epsilon) = -\infty. \tag{4.10}$$

A sequence of stochastic processes $\{\mathbf{X}_n\}_n$ that is exponentially tight in $\mathbb{D}(E)$ is C-exponentially tight if for each $\eta > 0$ and $T > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\sup_{s\leq T} d(\mathbf{X}_n(s), \mathbf{X}_n(s-)) \geq \eta) = -\infty. \tag{4.11}$$

Then Theorem 4.13 of [11] gives: an exponentially tight sequence $\{\mathbf{X}_n\}_n$ in $\mathbb{D}(E)$ is C-exponentially tight if and only if each rate function I that gives the LDP for a subsequence $\{\mathbf{X}_{n(k)}\}_{n(k)}$, satisfies $\mathsf{I}(\mathbf{x}) = \infty$ for each $\mathbf{x} \in \mathbb{D}(E)$ such that $\mathbf{x} \notin \mathbb{C}(E)$.

The stochastic assumptions of Section 2.2 ensure that the sequence of processes $\{\mathbf{Net}_n\}_n$ satisfies a LDP with good rate function (this implies that the sequence is exponentially tight) giving an infinite mass to discontinuous path. Hence the sequence of processes $\{\mathbf{Net}_n\}_n$ is C-exponentially tight.

We have to show that the sequence of processes $\{(\mathbf{A}_n, \mathbf{D}_n)\}_n$ is exponentially tight. The fact of dealing with non-decreasing processes simplifies the definitions. For $\mathbf{X} \in \mathbb{D}(\mathbb{R}_+^K)$ (or $\mathbb{D}(\mathbb{M}^K)$) non-decreasing, $\delta > 0$ and $T > 0$, we define $w_\delta(\mathbf{X}, T) = \sup_{t\in[0,T]} \|\mathbf{X}(t+\delta) - \mathbf{X}(t)\|$. We have clearly $w'(\mathbf{X}, \delta, T) = w_\delta(\mathbf{X}, T)$ and if $\{\mathbf{X}_n(0)\}_n$ is exponentially tight then (4.10) implies that $\{\mathbf{X}_n(t)\}_n$ is exponentially tight for each $t > 0$. Lemmas 5.1 and 5.2 show that conditions (4.10) and (4.11) are satisfied for the sequence of processes $\{(\mathbf{A}_n, \mathbf{D}_n)\}_n$. The exponential tightness of $\{(\mathbf{A}_n(0), \mathbf{D}_n(0))\}_n$ is clear since $\mathbf{A}_n(0) = \mathbf{D}_n(0) = 0$.

4.3. **Large deviations results.**

Proposition 4.1. *The sequence of processes $\{(\mathbf{A}_n, \mathbf{D}_n)\}_n$ satisfies a LDP in $\mathbb{D}(\mathbb{R}_+^K) \times \mathbb{D}(\mathbb{R}_+^K)$ with good rate function $\mathsf{I}^{\mathbf{A},\mathbf{D}}$. For $\mathbf{A}, \mathbf{D}$ absolutely continuous and such that $\mathbf{A}(0) = \mathbf{D}(0) = 0$ and $\mathbf{A} \geq \mathbf{D}$, $\mathsf{I}^{\mathbf{A},\mathbf{D}}$ is given by*

$$\mathsf{I}^{\mathbf{A},\mathbf{D}}(\mathbf{A}, \mathbf{D}) = \int_0^\infty H(\mathbf{A}(s), \mathbf{D}(s), \dot{\mathbf{A}}(s), \dot{\mathbf{D}}(s))ds, \tag{4.12}$$

where $H(A, D, \dot{A}, \dot{D}) := \inf_{P,N} h(A, D, \dot{A}, \dot{D}, P, N)$, with h given by,

$$h(A, D, \dot{A}, \dot{D}, P, N) := \sum_{i\in E(A,D)} \mathsf{I}^{\mathbf{S}^{(i)}}(\dot{D}^{(i)})\mathbb{1}_{\{\dot{D}^{(i)}>\mu^{(i)}\}} + \sum_{i\notin E(A,D)} \mathsf{I}^{\mathbf{S}^{(i)}}(\dot{D}^{(i)}) + \sum_i \dot{D}^{(i)}\tilde{\mathsf{D}}(P^{(i)}\|R^{(i)}) + \mathsf{I}^{\mathbf{N}}(N)$$

where $E(A, D) = \{i,\ A^{(i)} = D^{(i)}\}$ and with the infimum taken over the set of $(P, N) \in \mathbb{M}^K \times \mathbb{R}_+^K$ such that

$$\dot{A} = N + P^t\dot{D}.$$

For all other $\mathbf{A}, \mathbf{D}$, we have $\mathsf{I}^{\mathbf{A},\mathbf{D}}(\mathbf{A}, \mathbf{D}) = \infty$.

Proof. Thanks to the results of previous sections, conditions of Proposition 1.1 are satisfied and we define

$$\tilde{\mathsf{I}}^{\mathbf{A},\mathbf{D}}(\mathbf{A}, \mathbf{D}) = \inf\left\{\lim_{n\to\infty} \mathsf{I}^{\mathbf{Net}}(\mathbf{Net}_n),\ \{\mathbf{Net}_n\}_n \in \mathcal{S}(\mathbf{A}, \mathbf{D})\right\}, \tag{4.13}$$

where we recall that $\mathcal{S}(\mathbf{A}, \mathbf{D}) = \cup_{\mathbf{Net}}\mathcal{S}(\mathbf{Net}, \mathbf{A}, \mathbf{D})$, and $\mathcal{S}(\mathbf{Net}, \mathbf{A}, \mathbf{D})$ is defined in Proposition 4.1. We have to show that $\tilde{\mathsf{I}}^{\mathbf{A},\mathbf{D}} = \mathsf{I}^{\mathbf{A},\mathbf{D}}$ given by (4.12).

Consider $\mathbf{Net} \in \mathcal{D}_{\mathbf{Net}}$ and let $(\mathbf{A}, \mathbf{D}) = \Psi(\mathbf{Net})$. Let $\tau = \{0 = t_0 < t_1 < \dots\}$ be such that the processes $\mathbf{A}, \mathbf{D}, \mathbf{S}, \mathbf{N}$ and $\mathbf{D} \circ \mathbf{P}$ have a constant derivative on each (t_k, t_{k+1}). Then from $\mathbf{A} = \Gamma(\mathbf{D}, \mathbf{Net})$, we derive

$$\dot{\mathbf{A}}^{(i)}(t) = \dot{\mathbf{N}}^{(i)}(t) + \sum_j \dot{\mathbf{D}}^{(j)}(t)\dot{\mathbf{P}}^{(j,i)}(\mathbf{D}^{(j)}(t)).$$

From $\mathbf{D} = \Phi(\mathbf{A}, \mathbf{S})$, we get the following constraints:

- if $\mathbf{A}^{(i)}(t_k) > \mathbf{D}^{(i)}(t_k)$ or $\mathbf{A}^{(i)}(t_{k+1}) > \mathbf{D}^{(i)}(t_{k+1})$, then we have $\dot{\mathbf{D}}^{(i)}(t) = \dot{\mathbf{S}}^{(i)}(t)$ for $t \in (t_k, t_{k+1})$;
- otherwise $\mathbf{A}^{(i)}(t) = \mathbf{D}^{(i)}(t)$ for $t \in (t_k, t_{k+1})$ and we have $\dot{\mathbf{S}}^{(i)}(t) \geq \dot{\mathbf{A}}^{(i)}(t) = \dot{\mathbf{D}}^{(i)}(t)$ for $t \in (t_k, t_{k+1})$.

Now we can compute $\mathsf{I}^{\mathbf{Net}}(\mathbf{Net})$ as follows

$$\begin{aligned}\mathsf{I}^{\mathbf{Net}}(\mathbf{Net}) &= \int_0^\infty \sum_{i\in E(A,D)} \mathsf{I}^{\mathbf{S}^{(i)}}(\dot{\mathbf{S}}^{(i)}(s)) + \sum_{i\notin E(A,D)} \mathsf{I}^{\mathbf{S}^{(i)}}(\dot{\mathbf{D}}^{(i)}(s)) + \mathsf{I}^{\mathbf{N}}(\dot{\mathbf{N}}(s))ds \\ &\quad + \int_0^\infty \sum_j \dot{\mathbf{D}}^{(j)}(s)\tilde{\mathsf{D}}(\dot{\mathbf{P}}^{(j)}(s)\|R^{(j)})ds \\ &\geq \int_0^\infty h(\mathbf{A}(s),\mathbf{D}(s),\dot{\mathbf{A}}(s),\dot{\mathbf{D}}(s),\dot{\mathbf{P}}(s),\dot{\mathbf{N}}(s))ds \geq \mathsf{I}^{\mathbf{A},\mathbf{D}}(\Psi(\mathbf{Net})),\end{aligned}$$

since for $i\in E(\mathbf{A}(s),\mathbf{D}(s))$, we have $\mathsf{I}^{\mathbf{S}^{(i)}}(\dot{\mathbf{S}}^{(i)}(s)) \geq \mathsf{I}^{\mathbf{S}^{(i)}}(\dot{\mathbf{D}}^{(i)}(s))\mathbb{1}_{\{\dot{\mathbf{D}}^{(i)}(s)>\mu^{(i)}\}}$ because $\dot{\mathbf{S}}^{(i)}(s)\geq \dot{\mathbf{D}}^{(i)}(s)$ and $\mathsf{I}^{\mathbf{S}^{(i)}}$ is non-negative, convex with $\mu^{(i)}$ as unique zero. Hence, we have $\tilde{\mathsf{I}}^{\mathbf{A},\mathbf{D}} \geq \mathsf{I}^{\mathbf{A},\mathbf{D}}$.

Consider now $(\mathbf{A},\mathbf{D})$ such that $\mathsf{I}^{\mathbf{A},\mathbf{D}}(\mathbf{A},\mathbf{D})<\infty$, then we denote by $(\mathbf{p}(s),\mathbf{n}(s))$ the argument that achieves the minimum in $H(\mathbf{A}(s),\mathbf{D}(s),\dot{\mathbf{A}}(s),\dot{\mathbf{D}}(s))$ for any fixed s (note that h is a good rate function). Let $\mathbf{P}(\mathbf{D}(t))=\int_0^t \mathbf{p}(s)ds$ and $\mathbf{N}(t)=\int_0^t \mathbf{n}(s)ds$, note that $\mathbf{p}$ and $\mathbf{n}$ are measurable since H is a good rate function. We have $\mathbf{A}=\Gamma(\mathbf{D},\mathbf{P},\mathbf{N})$. Now define $\mathbf{s}(s)$ as follows:

- if $\mathbf{A}^{(i)}(s)=\mathbf{D}^{(i)}(s)$ then $\mathbf{s}^{(i)}(s)=\dot{\mathbf{D}}^{(i)}(s)\vee\mu^{(i)}$;
- if $\mathbf{A}^{(i)}(s)>\mathbf{D}^{(i)}(s)$ then $\mathbf{s}^{(i)}(s)=\dot{\mathbf{D}}^{(i)}(s)$.

We have $\mathbf{D}=\Phi(\mathbf{A},\mathbf{S})$ with $\mathbf{S}(t)=\int_0^t \mathbf{s}(s)ds$. Hence we have $(\mathbf{A},\mathbf{D})=(\Gamma(\mathbf{D},\mathbf{Net}),\Phi(\mathbf{A},\mathbf{Net}))$ for $\mathbf{Net}=(\mathbf{S},\mathbf{P},\mathbf{N})$ and $\mathsf{I}^{\mathbf{Net}}(\mathbf{Net})=\mathsf{I}^{\mathbf{A},\mathbf{D}}(\mathbf{A},\mathbf{D})<\infty$ by construction. Hence the sequence $\mathcal{S}(\mathbf{Net},\mathbf{A},\mathbf{D})=\{\mathbf{Net}_n\}_n$ is well-defined and we have $\tilde{\mathsf{I}}^{\mathbf{A},\mathbf{D}}(\mathbf{A},\mathbf{D})\leq \lim_{n\to\infty}\mathsf{I}^{\mathbf{Net}}(\mathbf{Net}_n)=\mathsf{I}^{\mathbf{Net}}(\mathbf{Net})=\mathsf{I}^{\mathbf{A},\mathbf{D}}(\mathbf{A},\mathbf{D})$. □

From this proposition, it is quite easy to derive a LDP for the process $\mathbf{Q}_n(t):=\mathbf{A}_n(t)-\mathbf{D}_n(t)$ counting the number of customers in each queue. Thanks to the contraction principle, we have

$$\mathsf{I}^{\mathbf{Q}}(\mathbf{Q})=\inf\{\mathsf{I}^{\mathbf{A},\mathbf{D}}(\mathbf{A},\mathbf{D}),\mathbf{Q}=\mathbf{A}-\mathbf{D}\},$$

which gives directly Theorem 2.1.

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5. Appendix

5.1. Properties of the map Γ and Φ.

For $\mathbf{X}\in\mathbb{D}(\mathbb{R}_+^K)$, $\delta>0$ and $T>0$, we define $w_\delta(\mathbf{X},T)=\sup_{t\in[0,T]}\|\mathbf{X}(t+\delta)-\mathbf{X}(t)\|$.

Lemma 5.1. *We have*

$$w_\delta(\Phi(\mathbf{X},\mathbf{Y}),T)\leq w_\delta(\mathbf{Y},T).$$

Proof. It is clearly sufficient to consider the case $K=1$. We will prove that

$$\Phi(\mathbf{X},\mathbf{Y})(t+\delta)-\Phi(\mathbf{X},\mathbf{Y})(t)\leq \mathbf{Y}(t+\delta)-\mathbf{Y}(t), \tag{5.1}$$

from which the lemma follows. If $\Phi(\mathbf{X},\mathbf{Y})(t)=\mathbf{Y}(t)$, then we have $\Phi(\mathbf{X},\mathbf{Y})(t+\delta)\leq\mathbf{Y}(t+\delta)$ and (5.1) is clear.

Assume now that $\Phi(\mathbf{X},\mathbf{Y})(t)=\inf_{0\leq s<t}\{\mathbf{Y}(t)-\mathbf{Y}(s)+\mathbf{X}(s)\}<\mathbf{Y}(t)$. We have

$$\mathbf{Y}(t+\delta)-\mathbf{Y}(s)+\mathbf{X}(s)=\mathbf{Y}(t)-\mathbf{Y}(s)+\mathbf{X}(s)+\mathbf{Y}(t+\delta)-\mathbf{Y}(t),$$

and (5.1) follows by taking the minimum in $s \in [0,t]$ and observing that $\Phi(\mathbf{X},\mathbf{Y})(t+\delta) \leq \inf_{0\leq s<t}\{\mathbf{Y}(t+\delta)-\mathbf{Y}(s)+\mathbf{X}(s)\}$. □

The following lemma is clear:

Lemma 5.2. *We have*

$$w_\delta(\Gamma(\mathbf{X},\mathbf{P},\mathbf{N}),T) \leq w_\delta(\mathbf{N},T) + w_\delta(\mathbf{P},\|\mathbf{X}(T)\|).$$

Lemma 5.3. *Assume $\mathbf{S} \in \mathbb{D}_0(\mathbb{R}_+)$ is absolutely continuous, then for any $\mathbf{A} \in \mathbb{D}(\mathbb{R}_+)$, we have $\mathbf{D} := \Phi(\mathbf{A},\mathbf{S})$ is absolutely continuous and,*

- *for all t such that $\mathbf{A}(t) > \mathbf{D}(t)$, we have $\dot{\mathbf{D}}(t) = \dot{\mathbf{S}}(t)$;*
- *if $\mathbf{A}(t) = \mathbf{D}(t)$ for $t \in (u,v)$ with $u < v$, then we have $\dot{\mathbf{S}}(t) \geq \dot{\mathbf{A}}(t) = \dot{\mathbf{D}}(t)$ for $t \in (u,v)$.*

Proof. It follows directly form (5.1) that if $\mathbf{S}$ is absolutely continuous, then $\Phi(\mathbf{X},\mathbf{S})$ is absolutely continuous for any $\mathbf{X}$. The rest of the lemma is obvious. □

5.2. Auxiliary results.

Lemma 5.4. *Given a substochastic matrix R such that $\rho(R) < 1$ and a substochastic matrix P such that the support of P is included in the support of R, i.e. $R^{(i,j)} = 0 \Rightarrow P^{(i,j)} = 0$. Then for any ϵ such that $0 < \epsilon^{(i)} \leq 1$ for all i, the matrix with coefficients $M^{(i,j)} = (1-\epsilon^{(i)})P^{(i,j)} + \epsilon^{(i)}R^{(i,j)}$ is of spectral radius less than 1.*

Proof. By a suitable permutation of rows and columns, we can assume that R is given in its canonical form

$$R = \begin{pmatrix} S_1(R) & * & * & * \\ 0 & S_2(R) & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & S_n(R) \end{pmatrix}, \tag{5.2}$$

where each $S_i(R)$ is an irreducible matrix. We have $\rho(R) < 1$ if and only if each $S_i(R)$ is not a stochastic matrix.

In view of the assumption on the support of P, the matrix P has the same structure as (5.2) and we have with the same notation as above, $S_i(M)$ which is an irreducible and not stochastic matrix. □

5.3. An example.

5.3. An example. In this section, we construct 2 different sequences of Jackson networks $\mathbf{Net}_n^1$ and $\mathbf{Net}_n^2$ such that their fluid limits are the same

$$\mathbf{Net}_n^1 \to \mathbf{Net} \quad \text{and} \quad \mathbf{Net}_n^2 \to \mathbf{Net},$$

but such that

$$\begin{aligned} (\mathbf{A}_n^1,\mathbf{D}_n^1) = \Psi(\mathbf{Net}_n^1) &\quad\to\quad (\mathbf{A}^1,\mathbf{D}^1), \\ (\mathbf{A}_n^2,\mathbf{D}_n^2) = \Psi(\mathbf{Net}_n^2) &\quad\to\quad (\mathbf{A}^2,\mathbf{D}^2), \end{aligned}$$

with $(\mathbf{A}^1,\mathbf{D}^1) \neq (\mathbf{A}^2,\mathbf{D}^2)$.

We consider a toy example with only one station (hence we omit the superscript $.^{(1)}$ that refers to that only station). Once a customer is served, he can either go out of the network or go back to this same node. We define the following driving sequences:

$$\begin{aligned} T^n &= (\underbrace{1,\dots,1}_{n}, n, \underbrace{1,\dots,1}_{n}, n, \dots), \\ \sigma^n &= \alpha(1,1,\dots), \end{aligned}$$

with $\alpha < 1$. We define now two different routing sequences

$$\begin{aligned} \nu^n &= (\underbrace{2,\dots,2}_{n+1}, \underbrace{1,\dots,1}_{n+1}, \dots), \\ \nu^n(x) &= (\underbrace{2,\dots,2}_{\lfloor xn \rfloor}, 1, \underbrace{2,\dots,2}_{n-\lfloor xn \rfloor}, \underbrace{1,\dots,1}_{\lfloor xn \rfloor}, 2, \underbrace{1,\dots,1}_{n-\lfloor xn \rfloor}, \dots), \end{aligned}$$

where $x < 1$. We denote by $\mathbf{Net}^1_n = \{\sigma^n, \nu^n, T^n\}$ and $\mathbf{Net}^2_n = \{\sigma^n, \nu^n(x), T^n\}$. $\nu^n(x)$ is obtained from ν^n by only interchanging a 1 and a 2. Hence we have

$$\mathbf{Net}^1_n \to \mathbf{Net} \quad \text{and} \quad \mathbf{Net}^2_n \to \mathbf{Net}.$$

Indeed the fluid network $\mathbf{Net}$ is given on Figure 1.

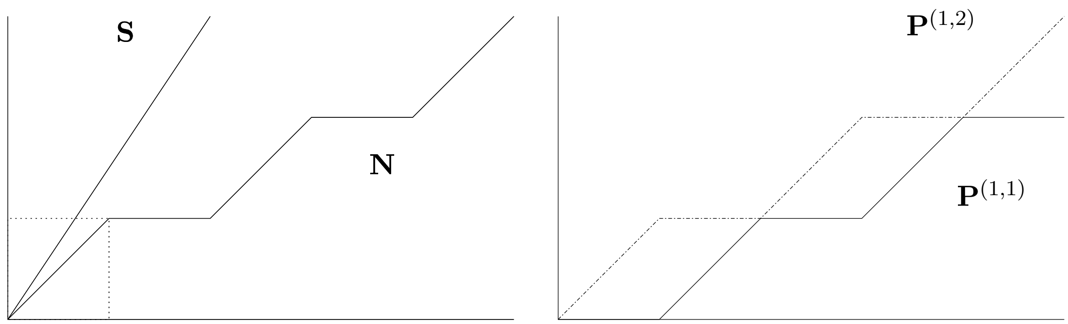

FIGURE 1. Fluid networks: **Net**

In the fluid limit, in case 1, the queue is always empty and the departure process is the same as the arrival process $\mathbf{N}$. In case 2, the fluid limit of the departure process and the queue length process is given on Figure 2.

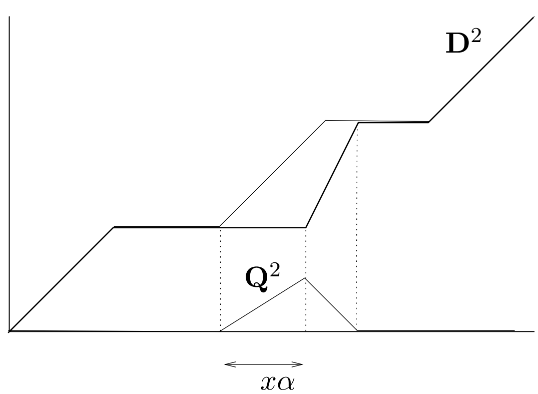

FIGURE 2. Departure process and queue length process

To explain $\mathbf{D}^2$, we write for each arrival (number on the left) the couple corresponding to: the inter-arrival time | the routing decision (1 means that the customer goes back in the queue and 2 means that the customer leaves the network):

$$
\begin{array}{ccccc}
1 & \to & 1 & | & 2 \\
2 & \to & 1 & | & 2 \\
3 & \to & 1 & | & 2 \\
 & & \vdots & & \\
\lfloor xn \rfloor & \to & 1 & | & 2 \\
\lfloor xn \rfloor + 1 & \to & 1 & | & 1,2 \\
\lfloor xn \rfloor + 2 & \to & 1 & | & 2 \\
 & & \vdots & & \\
n & \to & 1 & | & 2 \\
n+1 & \to & n & | & \underbrace{1,\dots,1}_{\lfloor xn \rfloor},2 \\
n+2 & \to & 1 & | & \underbrace{1,\dots,1}_{n-\lfloor xn \rfloor},2 \\
n+3 & \to & 1 & | & 2 \\
 & & \vdots & &
\end{array}
$$

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